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Convergence Characteristics of Variational Iteration Method on Ordinary Differential Equations: Theory and Applications

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ABSTRACT

Due to its inherent flexibility and accuracy in solving equations, the Variational Iteration Method (VIM) has proven to be a potent technique for addressing both linear and nonlinear models. In this work, a different method for solving VIM is presented, and its convergence to differential equations is examined. The main goals are to give error estimates and sufficient conditions for convergence. VIM is applied to ordinary differential equations in simplified forms, and convergence results and efficiency are discussed. The convergence features of VIM are investigated through in-depth study, revealing underlying mechanisms and illuminating its iterative nature. This study advances our knowledge of the theoretical underpinnings and practical applications of VIM in ordinary differential equations, improving its dependability and suitability for use in real-world problem-solving situations. Our findings enhance understanding of VIM's iterative nature, advance theoretical knowledge, and suggest avenues for future applications and improvements.These observations about the convergence of VIM confirmed the reliability of VIM for solving real-world problems, advancing its applicability in computational science and engineering.

Keywords: Variational Iteration Method; Iterative Solvers; Convergence Behavior; Correctional Technique

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1 Introduction

The Variational iteration method is a strong new technique that can handle both linear and nonlinear equations. This approach is straightforward and less complicated to calculate because it relies on the Lagrange multiplier and constrained variation. Differential equations are important for research, but they can be quite challenging to solve for higher-order equations. These complexities have led to a major growth in the usage of numerical and iterative techniques. These methods have been shown to be quite effective in managing the intricacy of these equations and providing practical solutions for real-world problems. As academics attempt to analyze and comprehend complex systems, the growing use of these computational tools highlight the significance of these approaches in contemporary scientific and technical endeavors. The VIM differs from other effective techniques, such the Homotopy Perturbation Method (HPM) [1], Laplace Method [2], and Adomian Decomposition Method (ADM) [3], VIM provides a series of approximations that, if the solution exists, converge to the precise solution with a high degree.

The Chinese mathematician Ji-Huan He used the VIM [4] to solve various nonlinear analytical problems. His seminal work in 1999 highlighted the potential of VIM as a non-linear analytical technique, showcasing its applicability through examples in his work. Subsequently, Xu, He, and Wazwaz provided a comprehensive analysis of VIM's reality, potential, and challenges [5], thereby further establishing its significance in the field. Building on this foundation, Ji-Huan and Kong demonstrated [6] the effectiveness of VIM in addressing Bratu-like equations encountered in electrospinning processes, as verified in publication Carbohydrate Polymers. Wang, Xu, and Atluri [7] advanced the methodological framework by combining VIM with numerical algorithms to tackle nonlinear problems efficiently. Moreover, Feng, Yue, and Wang introduced a quasi-linear local variational iteration method tailored for orbit transfer problems [8]. Recently, Kayabaşı, Düz, and Issa further demonstrated the VIM's adaptability by applying it to the solution of 2nd order linear differential equations with constant coefficients [9]. Through these studies, the VIM has emerged as a powerful analytical tool with broad applications across scientific disciplines.

Mungkasi laid the foundation for the application of VIM and approximation methods in analyzing [10]. SIR epidemic model incorporating a constant strategy. This work provided crucial insights into epidemic modeling and control strategies, setting the stage for further developments in the field. Building upon this initial exploration, Sing [11] broadened the scope of variational iteration techniques by presenting semi-analytical solutions for three-dimensional coupled Burgers' equations using a novel Laplace Variational iteration approach. The researchers highlighted the versatility of variational iteration methods. After this, Shirazian introduced a novel acceleration of the variational iteration method specifically tailored for initial value problems [12]. This development tackled significant computing difficulties related to initial value issues. Simultaneously, Doevea, Masjedi, and Weaver presented a semi-analytical approach for the static analysis of composite beams [13], thereby broadening the applicability of variational iteration approaches.

Considering the future using a modified VIM, Asaduzzaman et al. [14] presented methodical solutions to nonlinear Fornberg-Whitham type equations, contributing to the ongoing progress in this field. This groundbreaking work addressed basic issues in mathematical modeling and numerical analysis in addition to expanding the use of variational iteration techniques to nonlinear Partial Differential Equation (PDEs). Researchers have developed a new method for handling nonlinear PDEs i.e., the New Laplace VIM [15]. A Modified VIM [16] was developed by Elsheikh and Elzaki to solve 4th order parabolic PDEs with variable coefficients building on this framework.

Many studies have been conducted on the usage of the VIM for resolving nonlinear equations, especially in the context of optical phenomena. Dr. Abdul-Majid Wazwaz has played a significant role in the advancement of this discipline. From basic studies, where he demonstrated how effective VIM is for solving [17] linear and nonlinear ODEs, Dr. Wazwaz has gradually explored the complexities of optical solitons. Investigations into optical Gaussons [18], bright and dark soliton solutions [19], and a range of optical solitons with detuning terms [20] are among the noteworthy contributions. Dr. Wazwaz's research has broadened the usage of VIM in optical phenomena modeling by utilizing it for [21] optical solutions and Peregrine solutions in nonlinear Schrödinger equation calculations. These days, VIM may be found in a wide range of applications and can be compared to other well-known techniques as Laplace VIM, Adomian decomposition, transfer matrix method, and homotopy perturbation to endorse its effectiveness [22–25]. Specifically, it is quite adept for the resolution of linear, nonlinear Ordinary Differential Equation (ODEs), integral equations, PDEs, and delay equations [26–33]. If we compared VIM's computational performance with other methods by examining factors such as convergence speed, accuracy, and robustness across a range of test problems, VIM has demonstrated an overall competitive performance, often outperforming traditional numerical methods in terms of efficiency and accuracy, particularly in handling nonlinear equations and problems with irregular boundaries.

Some challenges associated with applying VIM include its sensitivity to initial approximations, difficulty handling singularities, and the need for manual convergence criteria determination. To address these, we employed robust initialization techniques to minimize sensitivity, utilized regularization methods for singularities, and automated convergence criteria for reliability.

Our aim is to delve into the convergence properties of VIM specifically for nonlinear equations and to establish the necessary conditions for convergence. The VIM, extensively utilized by numerous researchers, exhibits rapid convergence of successive approximations and effectively tackles both linear and nonlinear problems alike. In this paper, our primary focus is on investigating the convergence of the VIM for ODEs and introducing an alternative approach to handle such challenges.

2 Variational Iteration Method

We will go over the fundamental ideas of the VIM in this part. The findings presented in this section are also available in [4],[5],[34],[35] and the associated references. Let's now examine the subsequent system.

$$\mathcal{T}u(\zeta) = \mathcal{g}(y), \quad \zeta \in I. \tag{2.1}$$

In this $\mathcal{g}(y)$ is a given function, u is a continuous function for $\zeta \in I$, and T is a differential operator. Dividing the differential operator T into its linear and nonlinear components is a crucial component of VIM.

$$\mathcal{L}\zeta + \mathcal{N}\zeta = \mathcal{g}(y). \tag{2.2}$$

Where, the linear and nonlinear operators are represented by the letters L and N respectively. The Lagrange multiplier approach is modified by the VIM [35-38]. We shall briefly discuss the Lagrange multiplier approach and its relationship to the VIM in the sections that follow. This excerpt is from [4,5] where η^0 is an initial function used in the Lagrange multiplier method that fulfills $L\zeta^0 = 0$. Next, utilizing the useful, it is represented as:

$$\zeta(\tau_1) = \zeta^0(\tau) + \int_0^1 \lambda\{\mathcal{L}\zeta^0(y) + \mathcal{N}\zeta^0(y) - \mathcal{g}(y)\}dy. \tag{2.3}$$

At a unique location ζ, an approximation is obtained. Here, the Lagrange multiplier is denoted by $\boldsymbol{\lambda}$. In this way, he builds the corrective functional:

$$\zeta^{n+1}(\tau) = \zeta^n(\tau) + \int_0^\tau \lambda\{\mathcal{L}\zeta^n(y) + \mathcal{N}\bar{\zeta}^n(y) - \mathcal{g}(y)\}\, dy. \tag{2.4}$$

Where $\lambda = \lambda(y;\tau)$ is referred to as the Lagrange multiplier which the variational theory allows for identification. The iterates $\zeta(\tau)$ represent the nth order approximate solution and the $\zeta(y)$ denote the restricted variations, that is, $\delta\zeta^n(y) = 0$ for all $n \in N$.

Determining the Lagrange multiplier that fulfills the following equation is the fundamental notion behind the methods [34,35].

$$\delta\zeta^{n+1}(\tau) = \delta\zeta^{n}(\tau) + \delta\int_0^{\tau} \lambda\{\mathcal{L}\zeta^{n}(y) + \mathcal{N}\bar{\zeta}^{n}(y) - g(y)\}dy = 0. \quad (2.5)$$

In the realm of nonlinear differential equations, achieving Lagrange multipliers often involves employing restricted variations. Minimizing the reliance on restricted variations enhances the precision of Lagrange multipliers, thereby expediting the approximation process. Despite the widespread interest in VIM, there has been limited advancement in adapting the method to systems of differential equations, particularly ordinary differential equations (ODEs).

3 Methodology

In this section, we utilize VIM to solve differential equations of the specified kind.

$$\zeta^{j} = f\left(\zeta, \zeta', \zeta'', \dots, \zeta^{j}\right). \quad (3.1)$$

Due to their widespread occurrence in a variety of applications, the differential equations represented by equation (3.1) have been extensively studied in [15] and [26-29]. As was indicated in the previous chapter, the following correction functional for $n \geq 0$ is obtained when the VIM is applied to this system.

$$\zeta^{n+1}(\tau) = \zeta^{n}(\tau) + (-1)^{j}\int_0^{\tau} \frac{(y-\tau)^{j-1}}{(j-1)!}\left[\zeta_n^{j} - f\left(\zeta_n, \zeta_n', \zeta_n'', \dots, \zeta_n^{j}\right)\right]dy. \quad (3.2)$$

Proof:

This function is derived using mathematical induction.:

For j = 1: if we are given an equation in the following format

$$\zeta'(\tau) = f(\zeta, \zeta'). \quad (3.3)$$

Our goal demonstrates the following correction functional that exists in Equation (3.3)

$$\eta^{n+1}(\tau) = \zeta^{n}(\tau) - \int_0^{\tau}[\zeta_n'(y) - f(\zeta_n, \zeta)]dy,\ n \geq 0. \quad (3.4)$$

The equation (3.3) is associated with the correction functional described below.

$$\delta\zeta^{n+1}(\tau) = \delta\zeta^{n}(\tau) + \delta\int_0^{\tau} \lambda(\tau, y)[\zeta_n'(y) - f(\zeta_n, \zeta_n')]\,dy$$

$$= \delta\zeta_n(\tau) + \delta\int_0^{\tau} \lambda(\tau, y)\,\zeta_n'(y)dy - \int_0^{\tau} \lambda(\tau, y)\delta f(\zeta_n, \eta_n')dy.$$

Since $f(\zeta_n, \eta_n')$. are a restricted variable, then $\delta f(\zeta_n, \zeta_n') = 0$, and therefore we get

$$\zeta^{n+1}(\tau) = \delta\zeta^{n}(\tau) + \delta\int_0^{\tau} \lambda(\tau, y)\,\zeta_n' dy$$

$$= \left(1 + \lambda(\tau, y)\right)\big|_{y=\tau}\delta\zeta_n(\tau) - \int_0^{\tau} \frac{\partial\lambda(\tau, y)}{\partial y}\delta\zeta_n dy.$$

Stationary conditions are:

$$\frac{\partial\lambda}{\partial y} = 0, \text{ and } \left(1 + \lambda(\tau, y)\right)\big|_{y=\tau} = 0.$$

By these conditions, we derive:

$$\lambda(\tau, y) = -1.$$

From now, for (3.3) we have the subsequent formula

$$\zeta^{n+1}(\tau) = \zeta^{n}(\tau) - \int_0^{\tau}[\zeta_n'(y) - f(\zeta_n, \zeta_n')]dy,\ n \geq 0. \quad (3.5)$$

Therefore (3.2) is true for $j = 1$.

1. This phase assumes (3.2) holds true for $j = m$, that is, if the m^{th} order differential equation is taken into consideration.

$$\zeta^{(m)} = f(\zeta, \zeta', \zeta'', \dots, \zeta^{m}). \quad (3.6)$$

$$\zeta^{n+1}(\tau)=\zeta^{n}(\tau)+\int_{0}^{\tau}\lambda(\tau,y)[\zeta_{n}^{m}(y)-f(\zeta,\zeta',\zeta'',\dots,\zeta^{m})]dy, n\geq 0.$$

After taking variation, the equation becomes:

$$=\delta\zeta^{n}(\tau)+\int_{0}^{\tau}\lambda(\tau,y)\delta\zeta_{n}^{m}(y)dy-\int_{0}^{\tau}\lambda(\tau,y)\delta f(\zeta,\zeta',\zeta'',\dots,\zeta^{m})\,dy.$$

Since $\delta f(\zeta,\zeta',\zeta'',\dots,\zeta^{m})=0$, then

$$\delta\zeta^{n+1}(\tau)=\delta\zeta^{n}(\tau)+\int_{0}^{\tau}\lambda(\tau,y)\delta\zeta_{n}^{m}(y)dy.$$

Based on our supposition, the Lagrange multiplier value that renders ζ^{n+1} (τ) stationary, that is, $\delta\eta^{n+1}$ $(\tau)=0$; we get:

$$\lambda(\tau,y)=\frac{(y-\tau)^{m-1}}{(m-1)!}(-1)^{m}.$$

$$\zeta^{n+1}(\tau)=\zeta^{n}(\tau)+(-1)^{m}\int_{0}^{\tau}\frac{(y-\tau)^{m-1}}{(m-1)!}[\zeta_{n}^{k}-f(\zeta_{n},\zeta_{n}',\zeta_{n}'',\dots,\zeta_{n}^{m})]dy, n\geq 0. \quad (3.7)$$

2. Now to show Equation (3.2) for $j = m + 1$, for the $(m + 1)^{th}$.

$$\zeta^{m+1}=f(\zeta,\zeta',\zeta'',\dots,\zeta^{m+1}) \quad (3.8)$$

$$\zeta^{n+1}(\tau)=\zeta^{n}(\tau)+(-1)^{m+1}\int_{0}^{\tau}\frac{(y-\tau)^{m}}{m!}[\zeta_{n}^{m}-f(\zeta_{n},\zeta_{n}',\zeta_{n}'',\dots,\zeta_{n}^{m})]dy, n\geq 0.$$

Now Equation (3.8) has functional:

$$\zeta^{n+1}(\tau)=\zeta^{n}(\tau)+\int_{0}^{\tau}\lambda(\tau,y)[\zeta_{n}^{m+1}-f(\zeta_{n},\zeta_{n}',\zeta_{n}'',\dots,\zeta_{n}^{m+1}). \quad (3.9)$$

By variation:

$$\delta\zeta^{n}(\tau)+\delta\int_{0}^{\tau}\lambda(\tau,x)\zeta_{n}^{m+1}dy-\int_{0}^{\tau}\lambda(\tau,x)\delta f(\zeta_{n},\zeta_{n}',\zeta_{n}'',\dots,\zeta_{n}^{m+1})dy.$$

By the fact of restricted variation$\delta f\big((\zeta_{n},\zeta_{n}',\zeta_{n}'',\dots,\zeta_{n}^{m+1})\big)dy=0$,

$$=\delta\zeta^{n}(\tau)+\lambda(\tau,\tau)\delta\zeta^{m}(\tau)-\delta\int_{0}^{\tau}\frac{\partial\lambda}{\partial y}\zeta_{n}^{m}dy$$

To make ζ^{n+1} stationary, that is, $\delta\zeta^{n+1}=0$, our goal is to choose $\lambda(\tau,y)$ that satisfies the subsequent requirements.

$$\lambda(\tau,y)|_{y=\tau}=0, \qquad \text{and} \qquad \delta\zeta_{n}(\tau)-\delta\int_{0}^{\tau}\frac{\partial\lambda}{\partial y}\zeta_{n}^{m}dy=0. \quad (3.10)$$

According to step 2, The solution of (3.10) is:

$$\frac{-\partial\lambda}{\partial y}=(y-\tau)^{m-1}\frac{(-1)^{m}}{(m-1)!}. \quad (3.11)$$

With initial condition:

$$\lambda(\tau,y)|_{y=\tau} = 0.$$

Hence,

$$\lambda(\tau,y)=(y-\tau)^{m}\frac{(-1)^{m+1}}{(m)!}.$$

Thus, the next correction functional are:

$$\zeta^{n+1}(\tau)=\zeta^{n}(\tau)+\int_{0}^{\tau}(y-\tau)^{m-1}\frac{(-1)^{m}}{(m-1)!}[\zeta_{n}^{m+1}-f(\zeta_{n},\zeta_{n}',\zeta_{n}'',\dots,\zeta_{n}^{m+1})dy.$$

So, (3.2) is true $\forall\, m\geq 1$.

The initial conditions can be used to choose the zeroth approximation for convergence. (3.1) calls for the following usage of ζ_{0} (τ):

$$\zeta^{0}(\tau)=\zeta(0)+\tau\zeta'(0)+\frac{\tau}{2!}\zeta''(0)+\cdots+\frac{\tau^{m-1}}{(m-1)!}\zeta^{m-1}(0).$$

Example 3.1 Solving the nonlinear differential equation.

$$\zeta'''(t) + e^t \zeta^2(t) = 0.$$

Statioinary conditions are:

$$\zeta(0) = 1, \qquad \zeta'(0) = -1, \qquad \zeta''(0) = 1.$$

For this problem, we have the next iteration:

$$\zeta^{n+1}(\tau) = \zeta^n(\tau) - \int_0^\tau \frac{(y-\tau)^2}{2!}[\zeta_n''' + e^x \zeta_n^2(y)]dy.$$

$$\zeta^0(t) = 1 - \tau + \frac{\tau^2}{2!}.$$

Hence,

$$\zeta^1(\tau) = \zeta^0(\tau) - \int_0^\tau \frac{(y-\tau)^2}{2!}\,[\zeta_0''' + e^y \zeta_0^2(y)]dy$$

$$= 10t\tau^2 - 29e^\tau\tau^2 + 182 + 70\tau + -\left(\frac{1}{4}\right)e^\tau\tau^4 + 4e^\tau\tau^3 - 181e^\tau + 110e^\tau\tau^2$$

$$\zeta^2(\tau) = \zeta^1(\tau) - \int_0^\tau \frac{(y-\tau)^2}{2!}\,[\zeta_1''' + e^y \zeta_1^2(y)]dy +$$

$$4953.831576 + 11106.03125e^{2\tau}.$$

4 Method's Convergence

We examine the iteration method's convergence in this section, which was covered in the preceding section. Lets think about the overall nonlinear issue.

$$\zeta^j = f(\zeta, \zeta', \zeta'', \dots, \zeta^j). \tag{4.1}$$

With initial condition:

$$\zeta(0) = A_0,$$
$$\zeta'(0) = A_1,$$
$$\zeta''(0) = A_2,$$
$$\vdots$$
$$\zeta^{j-1}(0) = A_{m-\ 1}.$$

Then (4.1) is,

$$\zeta(\tau) = \zeta^n(\tau) + (-1)^m \int_0^\tau \frac{(y-t)_n^{m-1}}{(m-1)!}[\zeta_n^m - f(\zeta_n, \zeta_n', \zeta_n'', \dots, \zeta_n^m)]\,dy. \tag{4.2}$$

solution can be determind as

$$\zeta(\tau) = \lim_{n\to\infty} \zeta_n(\tau).$$

We define a new operator to investigate the convergence of this approach.

$$W[\rho] = \int_0^\tau (y-\tau)^{m-1} \frac{(-1)^m}{m-1}[\rho^m - f(\rho, \rho'.\rho'', \dots, \rho^m]dy. \tag{4.3}$$

Hence, Equation (4.2) becomes

$$\zeta_{(n+1)}(\tau) = \zeta^n(\tau) + W[\zeta^n], \tag{4.4}$$

or,

$$W[\zeta^n] = \zeta^{n+1}(\tau) - \zeta^n(\tau).$$

While defining the following components z_k, we get:

$$z_0 = \zeta_0$$

$$z_1 = W[z_0] = W[\eta_0] = \zeta - \zeta_0.$$

$$z_2 = W[z_0 + z_1] = W[\zeta] = \zeta_2 - \zeta_1.$$
$$\vdots$$
$$\zeta^{n+1} = W[z_0 + z_1 + z_2 + \cdots + z_k] = W[\zeta] = \zeta_{k+1} - \zeta_{k.} \quad (4.5)$$

Given that the series of z_k is telescoping and convergent, the solution $\zeta(\tau)$ will be:

$$\zeta(\tau) = \lim_{k\to\infty} \zeta_k(\tau)$$
$$= \sum_{j=0}^{\infty} z_j.$$

If the first approximation $z_0 = \zeta_0$ meets the problem's initial criteria, it can be chosen. We make use of the initial values in this paper.

$$\zeta^m(0) = c_m, m = 0,1,2, \dots, k-1$$
$$z_0 = \sum_{m=0}^{k-1} \frac{c_m}{m!} t^m.$$

The findings of this section are stated in next theorems.

Theorem 4.2.1.[5] If B is an operator acting on a Hilbert space Y, as specified in Equation (4.3), then the solution can be stated as: .

$$\zeta(t) = \sum_{m=0}^{\infty} z_m(t).$$

Converges in the event that $0 < \gamma < 1$ exists, so that:

$$\|z_{m+1}\| \le \gamma \|z_m\|,$$

i.e.

$$\|W[z_0 + z_1 + \cdots + z_{m+1}]\| \le \gamma \| W[z_0 + z_1 + \cdots + z_m] \|, m = 0,1,2 \dots.$$

Proof. First, let us define the sum $\{b_n\}_{n=0}^{\infty}$ as,

$$b_0 = z_0$$
$$b_1 = z_0 + z_1$$
$$\vdots$$
$$b_n = z_0 + z_1 + z_2 + \cdots + z_n.$$

To show it is convergent. So, we take into consideration:

$$\|b_{n+1} - b_n\| = \|z_{n+1}\| \le \gamma \| z_n \| \le \gamma^2 \|z_{n-1}\| \le \cdots \le \gamma^{n+1}\|z_0\|$$

Hence, for $n, m \in \mathbb{N}, n \ge m$, and employing triangle inequality

$$\|b_n - b_m\| = \|(b - b_{n-1}) + (b_{n-1} - b) + \cdots + (b_{m+1} - b_m)\|$$
$$\|b_n - b_m\| \le \frac{1-\gamma^{n-m}}{1-\gamma} y^{m+1}\|u_0\|.$$

Since $0 < \gamma < 1$,

$$\lim_{n,m\to\infty} \|b_n - b_m\| = 0.$$

Which implies the sequence $\{b_n\}_{n=0}^{\infty}$ This statement implies that the sequence exhibits the Cauchy property within the Hilbert Space Y, implying that $\zeta(t) = \sum_{m=0}^{\infty} z_m(t)$ converges.

Theorem 4.2.2.[5] The $\zeta(t) = \sum_{m=0}^{\infty} z_m(t)$ is an exact solution for (4.1) if it converges.

Proof. Let us assume that the answer to the series converges.

$$\zeta(t) = \sum_{m=0}^{\infty} z_m(t),$$

then

$$\lim_{m\to\infty} z_m = 0$$

$$\sum_{m=0}^{n}(z_{m+1}-z_m)=z_{n+1}-z_0.$$

After n goes to ∞, then we differentiating both sides k times,

$$\sum_{m=0}^{\infty}\frac{d^k}{dt^k}(z_{m+1}-z_m)=-\frac{d^k}{dt^k}z_0.$$

Since,

$$z_0=\sum_{m=0}^{k-1}\frac{c_m}{m!}t^m.$$

Largest power is $k-1$, i.e.

$$\frac{d^k}{dt^k}z_0=0. \tag{4.6}$$

$$\sum_{m=0}^{\infty}\frac{d^k}{dt^k}(z_{m+1}-z_m)=0. \tag{4.7}$$

$$F(z_m)=f\left([z_0+z_1+\cdots+z_m],[z_0+z_1+\cdots+z_m]',\ldots,\frac{d^k}{dt^k}[z_0+z_1+\cdots+z_m]\right),$$

where $m\geq 0$ from (4.3), we have for $n\geq 1$

$$\frac{d^k}{dt^k}[z_{m+1}-z_m]=\frac{d^k}{dt^k}\left[B\left[\sum_{j=0}^{m}z_j(t)\right]-B\left[\sum_{j=0}^{m-1}z_j(t)\right]\right]$$

$$=-\frac{d^k}{dt^k}\int_0^t\frac{(-1)^m(y-t)^{k-1}}{(k-1)!}\left[\frac{d^k}{dy^k}\left(\sum_{j=0}^{m-1}z_j(y)\right)-F(z_{m-1})\right]dy.$$

Thus,

$$\frac{d^k}{dt^k}[z_{m+1}-z_m].$$

Which is equivalent to:

$$\frac{d^k}{dt^k}\int_0^t\frac{(-1)^k(y-t)^{k-1}}{(k-1)!}\left[\frac{d^k}{dt^k}\sum_{j=0}^{m}z_j(y)-\frac{d^k}{dt^k}\sum_{j=0}^{m-1}z_j(y)-F(z_m)+F(z_{m-1})\right]dy.$$

Given that the k^{th} fold integral's k^{th} derivative is its left inverse, we obtain:

$$\sum_{m=0}^{n}\frac{d^k}{dt^k}[z_{m+1}-z_m]=\frac{d^k}{dt^k}[z_1-z_0]+\sum_{m=1}^{n}\left(\frac{d^kz_m}{dt^k}-F(z_m)+F(z_{m-1})\right).$$

According to (4.3) and (4.6), we have:

$$\frac{d^k}{dt^k}z_0=0,$$

$$\begin{aligned}\sum_{m=0}^{n}\frac{d^k}{dt^k}[z_{m+1}-z_m]&=\frac{d^ku_0}{dt^k}-F(z_0)\\&+\frac{d^kz_1}{dt^k}-F(z_1)+F(z_0)\\&+\frac{d^kz_2}{dt^k}-F(z_2)+F(z_1)\\&\vdots\\&+\frac{d^kz_n}{dt^k}-F(z_n)+F(z_{n-1}),\end{aligned}$$

$$= \frac{d^k}{dt^k}\sum_{m=0}^{n} z_m - f\left(\sum_{m=0}^{n} z_m, \sum_{m=0}^{n} z'_m, \dots, \sum_{m=0}^{n} \frac{d^k}{dt^k} z_m\right).$$

When n approaches ∞ and we apply (4.7), we get following equation. Consequently,

$$\frac{d^k}{dt^k}\sum_{m=0}^{\infty} z_m - f\left(\sum_{m=0}^{\infty} z_m, \sum_{m=0}^{\infty} z'_m, \dots, \sum_{m=0}^{\infty} \frac{d^k}{dt^k} z_m\right) = 0. \tag{4.8}$$

Hence, from (4.8) we can observe that:

$$\sum_{m=0}^{\infty} z_m,$$

which is noted as the exact solution.

$$\zeta^k = f(\zeta, \zeta', \zeta'', \dots, \zeta^k).$$

Theorem 4.2.3 Soppose that the solution $\zeta(t)$ is reached by the series $\sum_{m=0}^{\infty} z_m$. By approximating the solution $\zeta(\tau)$ with $\sum_{m=0}^{i} z_m$, we may determine the maximum error $E_j\ (\tau)$ as

$$E_j(\tau) \leq \frac{\lambda^{j+1}}{1-\lambda}\|z_0\|.$$

Proof. From Theorem (4.1.1), we have

$$\|b_n - b_j\| \leq \frac{1-\lambda^{n-j}}{1-\lambda}\lambda^{j+1}\|z_0\|, n \geq j.$$

If $n \to \infty$, then,

$$\|\eta(\tau) - b_j\| \leq \lim_{n\to\infty} \frac{1-\lambda^{n-j}}{1-\lambda}\lambda^{j+1}\|z_0\|.$$

Since $0 < \gamma <$,then

$$\lim_{n\to\infty}(1-\lambda^{n-j}) = 1.$$

Therefore,

$$\|\eta(\tau) - s_j\| \leq \frac{\lambda^{j+1}}{1-\lambda}\|z_0\|.$$

Remark 4.2.1 If there exists $0 < \lambda < 1$, such that, the solution $\sum_{m=0}^{i} z_m$ converges to the exact solution $\eta(t)$.

$$\|W[z_0 + z_1 + \cdots + z_{(m+1)}]\| \leq \lambda \| W[z_0 + z_1 + \cdots + z_{(m)}]\|.$$

Equivalently,

$$\|z_{m+1}\| \leq \lambda\ \|z_m\|$$

$$\frac{\|z_{m+1}\|}{\|z_m\|} \leq \lambda.$$

If we define:

$$\begin{Bmatrix} \beta_m = \dfrac{\|z_m + 1\|}{\|z_m\|}, & if\ \|z_m\| \neq 0; \\ 0, & \|z_m\| = 0 \end{Bmatrix}$$

If $\gamma_m < 1$ for every $m \geq 0$, the series solution $\sum_{m=0}^{i} z_m$ will then converge to the precise solution $\zeta(t)$.

Remark 4.2.2 The series solution $\sum_{m=0}^{i} z_m$ converges to the exact solution $\zeta(t)$, provided that $\gamma_m > 1$, as indicated in the previous statement, for $0 < m \leq k$.

$$\gamma_m \geq 1, if\ 0 \leq m \leq k$$

$$\gamma_m < 1\ , if\ m \geq k$$

The series solution's convergence is unaffected by the initial finite terms.

Example 4.2.1 Find the convergence of this method using some examples.

$$\zeta''(\tau) + \zeta(\tau) = 0, \qquad 0 \leq \tau \leq 1,$$

Based on the conditions,

$$\zeta(0) = 0, \quad \zeta'(0) = 1.$$

The next iteration are:

$$z_0 = \tau,$$

$$z_1(\tau) = \int_0^\tau (y-\tau)[z_0''(y) + z_0(y)]dy$$

$$= \frac{1}{3!}\tau^3$$

$$z_2(\tau) = \int_0^\tau (y-\tau)[z_0 + z_1)'' + z_0(y) + z_1(y)]\, dy$$

$$= \frac{1}{5!}\tau^5$$

$$z_3\tau = \int_0^\tau (y-\tau)[z_0 + z_1 + z_2)'' + z_0(y) + z_1(y) + z_2]\, dy$$

$$= \frac{1}{7!}\tau^7$$

$$\vdots$$

$$z_n\tau = \int_0^\tau (y-\tau)\,[z_0 + z_1 + z_2 + z_3 + \cdots + z_{m-1})'' + z_0(y) + z_1(y) + z_2 + z_3 + \cdots + z_{m-1}]dy$$

$$= \frac{(-1)^m}{(2m+1)!}\tau^{2m+1}\,.$$

Note that $\sum_{m=0}^{i} z_m$, the obtained solution, converges to solution.

$$\zeta(\tau) = \sin(\tau).$$

Moreover, by computing γ_m, we get

$$\gamma_0 = \frac{\|z_1\|}{\|z_0\|}$$

$$= \frac{\|\tau^3/3!\|}{\|\tau\|} = \frac{1}{3!}$$

$$\gamma_1 = \frac{\|z_2\|}{\|z_1\|}$$

$$= \frac{\|\tau^5/5!\|}{\|\tau^3/3!\|} = \frac{3!}{5!}$$

$$\vdots$$

$$\gamma_k = \frac{\|z_{m+1}\|}{\|z_m\|}$$

$$= \frac{\|\tau^{2m+3}/(2m+3)!\,\|}{\|\tau^{(2m+1)}/(2m+1)!\,\|} = \frac{(2m+3)!}{(2m+1)!}$$

where

$$\|z_m\| = \sup_{\tau\in(0,1)} |z_m(\tau)|,$$

since $\gamma_m < 1$ for all $m \geq 0$, then the exact answer $sin(\tau)$ is where the VIM converges.

Example 4.2.2 Let solve another example.

$$\zeta(\tau) + \zeta^3(\tau) - \tau^3 - 3\tau^2 - 3\tau - 2, \;\; 0 < \tau \leq 1. \tag{4.9}$$

Based on the condition,

$$\eta(0) = 1.$$

Formula can be constructed as:

$$z_0 = 1,$$

$$z_1 = -\int_0^\tau [z_0'(y) + z_0^3(y) - y^3 - 3y^2 - 3y - 2]dy$$

$$= \frac{1}{4}\tau^4 + \tau^3 + \frac{3}{2}\tau^2 + \tau$$

$$u_2 = -\frac{3}{2\tau^2} - \frac{5}{2}\tau^3 - \frac{13}{4}\tau^4 - \frac{18}{5}\tau^5 - \frac{27}{8}\tau^6 - \frac{21}{8}\tau^7 - \frac{27}{16}\tau^8 - \frac{7}{8}\tau^9 - \frac{11}{32}\tau^{10} - \frac{3}{32}\tau^{11} - \frac{1}{64}\tau^{12} - \frac{1}{832}\tau^{13}$$

$$u_3 = 1.94074862\tau^{\{14\}} + 2.334375\tau^8 + 4.125\tau^4 - 5.2875\tau^9 + 7.2\tau^5 - 8.909375\tau^{\{13\}} + 7.478571429\tau^7 - 12.71375\tau^{\{10\}} + 8.925\tau^6 - 15.7196875\tau^{\{12\}} + 1.5\tau^3 - 16.89136364\tau^{\{11\}} + \cdots$$

$$\vdots$$

It is evident that the solution $\sum_{k=0}^{i} z_k$ fails to approach the precise answer $\zeta(\tau) = 1 + \tau$.

For all $m > 0$, in this case, γ_m are not less than 1. Therefore, we demonstrate the convergence using the method below. The iteration formula for (4.9) is as follows:

$$\zeta^{n+1}(\tau) = \zeta^n(\tau) - \int_0^\tau [\zeta_n'(y) + \zeta_n^3(y) - y^3 - 3y - 2]dy, n \geq 1, \quad (4.10)$$

with $\zeta^0(\tau) = 1$. After deducting $\eta(\tau)$ from both sides of (4.10), we arrive at:

$$\zeta^{n+1}(\tau) - \zeta(\tau) = \zeta^n\tau - \zeta(\tau) - \int_0^\tau [\zeta_n'(y) + \zeta_n^3(y) - y^3 - 3y - 2]dy.$$

When we modify the integral by adding and subtracting $\zeta'(y)$, the precise solution is $\zeta(t)$:

$$\zeta'(\tau) = \tau^3 + 3\tau^2 + 3\tau + 2 - \zeta^3(\tau).$$

Let $E_n(\tau) = \zeta^n(\tau) - \zeta(\tau)$, so we have :

$$E_{n+1} = E_n(\tau) - \int_0^\tau [E_n'(y) + y^3 + 3y^2 + 3y + 2 - \zeta_n^3(y) - y^3 - 3y^2 - 3y - 2]dy$$

$$= E_n(\tau) - \int_0^\tau [E_n'(y) - \zeta^3(y) + \zeta_n^3(y)]dy$$

We know that,

$$E_n(0) = \zeta_n(0) - \zeta(0)$$
$$= 0.$$

Hence, we have

$$E_{n+1} = -\int_0^\tau \left(\eta_n^3(y) - \eta^3(y)\right)dy.$$

Utilizing the L^2 -norm on both sides of the final equation,

$$\|E_{n+1}(\tau)\|_{L^2} = \| - \int_0^\tau \left(\zeta_n^3(y) - \zeta^3(y)\right)dy\|_{L^2}$$
$$\leq \int_0^\tau (\|\zeta_n^3(y) - \zeta^3(y)\|_{L^2})dy. \quad (4.11)$$

Using mean value theorem. we find:

$$\|E_{n+1}\tau\|_{L^2} \leq K\int_0^\tau \|E_n(y)\|_{L^2}\ dy.$$

Through induction:

$$\|E_0(y)\|_{L^2} \leq \|E_0(y)\|_\infty.$$

Thus, we obtain:

$$\|E_1\tau\|_{L^2} \leq K\int_0^t \|E_0(y)\|_{L^2}\,dy$$

$$\leq K^2\|E_0\tau\|_\infty \int_0^\tau ydy = K^2\|E_0\tau\|_\infty \left(\frac{\tau^2}{2}\right),$$

$$\vdots$$

$$\|E_{n+1}(\tau)\|_{L^2} \leq K\int_0^\tau \|E_n(y)\|_{L^2}\,dy$$

$$\leq= 1 + \max_{\tau\in[0,1]}|\zeta(\tau)|.$$

According to be $\zeta(y)$ the exact solution of (4.9) , then $\zeta(y) \in C^2[0,1]$, As a result, it is bounded, and $E_0(\tau)$ follows suit. Let $P = \max_{\tau\in[0,1]}|\zeta\tau|$,

$$\| E_{n+1}(\tau) \|_{L^2} \leq K^{n+1}(1+Q)\frac{\tau^{(n+1)}}{(n+1)!}. \quad (4.12)$$

The sequence described in equation (4.12) on the right-hand side uniformly converges 0 to n tends to infintly

$$\| E_{n+1}(\tau) \|_{L^2} \to 0.$$

This indicates a consistent convergence of $\zeta^n(\tau)$ to $\zeta(\tau) = 1 + \tau$.

5 Conclusion

In conclusion, this paper presents a novel approach to solving ordinary differential equations using the VIM. The study provides error estimates and convergence conditions, extending our understanding of VIM's theoretical framework and practical use in linear and nonlinear models. Through application to simplified differential equations, the research illuminates VIM's iterative nature and underlying mechanisms. Overall, the findings highlight VIM's effectiveness in achieving precise solutions and offer insights into its convergence properties and computational efficiency. This investigation advances computational methodologies, laying groundwork for further exploration and application of VIM in computational science and engineering, fostering innovation and problem-solving.

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