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Investigating a Coupled System of Mittag-Leffler Type Fractional Differential Equations with Coupled Integral Boundary Conditions

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ABSTRACT

In this work, A coupled system with Atangana Baleanu Caputo $(ABC)-$derivative under coupled integral boundary conditions is examined. The study aims to develop necessary and sufficient conditions for the existence and uniqueness of solution of the considered problem. In this connection, results for a minimum of one solution are obtained through the employment of Krasnoselskii's fixed point theorem. The Ulam-Hyers (U-H) concept is used to establish stability-related results. Finally, an example is given to illustrate the application of our found results.

Keywords: ABC Derivative; U-H Stability; Krasnoselskii's Fixed Theorem

1 Introduction

The last thirty years have seen a significant increase in the study of fractional calculus. For the reason that fractional differential systems and equations (FDEs) are crucial tools for describing physical processes that arise in physics, engineering, economics, and other disciplines. Comparing arbitrary order derivative to traditional integer order, it is amazing that the former is global in nature. This is one of the operator's many strengths. Because of these important applications, scholars have focused a lot of attention on examining a variety of real-world issues and occurrences that fall under the purview of the aforementioned calculus. In the realm of engineering [1,2], rheology [3], epidemiology [4], physical sciences [5], signal and image processing [6], etc., some intriguing work has been conducted recently. Scientists and researchers have focused on investigating FDEs in a variety of analyses, including stability, numerical, and qualitative theory, because of the significance of the aforementioned field. A lot of work has been published in this area. Only a few are relevant here, such as the fixed point theory's existence theory of FDEs [7], degree theory's qualitative results of FDEs [8], wavelet's numerical analysis of FDEs [9], numerical study of fractional drinking model [10], spectrum analysis of FDEs [11], decomposition technique for FDEs [12], and so on.

It is noteworthy that boundary value problems (BVPs) are widely used in mechanical engineering and mass heat transfer modelling of a variety of phenomena. Consequently, a great deal of research has been done in the area of BVPs under the umbrella of fractional calculus (see [13]). Very good research has also been done on BVPs of FDEs with integrals in their circumstances. Due to the fact that integral BVPs have many uses in practical domains such as chemical engineering, blood flow issues,

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population dynamics, subterranean water flow, and so on (see [14]). Remember that the aforementioned studies were examined using the standard Caputo or Reimann-Liouville fractional derivative. As is well known, there are many definitions of integral and fractional order differential operators, ranging from singular to non-singular. The aforementioned operators have been heavily utilised by researchers in various investigations. Here, we make reference to a few published works, including Mittag-Leffler type in [17], Caputo-Power law operator in [15], and Caputo-Fabrizio in [16]. The Caputo-Fabrizio and ABC operators have been used to a wide range of real-world problems in a number of published research works, as shown in [18–26]. The authors of all the aforementioned works have mostly addressed early value issues. The authors have concentrated on creating sufficient conditions for the presence of a solution to the given problem by applying nonlinear analysis tools. In this instance, we cite the research done on coupled systems with linked boundary conditions for the iterative solution by authors [27] and [28]. Similar to this, writers [29] have used a monotone iterative technique to study a linked system under coupled integral boundary conditions. However, all of the aforementioned studies were conducted using standard Caputo power-law derivatives.

Shah, et.al [30] studied a coupled system of coupled integral BVPs using Caputo-Fabrizio (CF) derivative. Since ABC derivative is the generalization of the aforementioned CF operator. Therefore, we extend the said coupled system to investigate under ABC operators with coupled integral boundary conditions for $\varkappa \in [0, T] = \mathscr{W}$ as follows:

$$\begin{cases} {}^{ABC}\mathbb{D}^{\imath_1}\mathrm{u}(\varkappa) = \mathrm{f}_1(\varkappa, \mathrm{u}(\varkappa), \mathrm{v}(\varkappa)), \;\; 0 < \imath_1 \leq 1, \\ {}^{ABC}\mathbb{D}^{\imath_2}\mathrm{v}(\varkappa) = \mathrm{f}_2(\varkappa, \mathrm{u}(\varkappa), \mathrm{v}(\varkappa)), \;\; 0 < \imath_2 \leq 1, \\ \mathrm{u}(0) = \int_0^T g_1(\mathrm{v}(s))ds, \\ \mathrm{v}(0) = \int_0^T g_2(\mathrm{u}(s))ds, \end{cases} \tag{1.1}$$

where $\mathrm{f}_1, \mathrm{f}_2 : \mathscr{W} \times R^2 \to R$, $g_1, g_2 : \mathscr{W} \to R$, are continuous functions, and $\phi, \psi \in L(\mathscr{W})$. The fixed point theorems of Banach and Krasnoselskii [31] are used to construct adequate conditions for the existence and uniqueness of a solution. Additionally, stability is an important tool that has been investigated recently very well for different problems in the sense of U-H type. Recently, authors [32], and [33] have used the U-H concept to study stability analysis of different problems with various fractional differential operators. To elaborate further, in the realm of mathematical analysis and specifically within the study of functional equations and dynamic systems, the concept introduced by Ulam-Hyers plays a pivotal role in examining the stability of various mathematical models. This notion, primarily dealing with the stability of functional equations, suggests a methodology to determine if slight alterations in the initial conditions of an equation will lead to small changes in the outcomes, hence ensuring the equation's stability. The (U-H) idea originating from the collaborative insights of Stanislaw Ulam and Donald Hyers, has been instrumental in bridging the gap between abstract mathematical theories and their practical applications in physics, engineering, and beyond. By utilizing this concept, researchers and mathematicians are able to provide rigorous proofs and frameworks that affirm the resilience and reliability of mathematical models under minor perturbations. This is especially crucial in areas where mathematical precision plays a fundamental role in predicting and shaping outcomes, such as in the simulation of physical systems, optimization problems, and the analysis of control systems. The implementation of the (U-H) idea to establish stability-related outcomes involves a meticulous process. It requires the identification of the specific conditions under which a functional equation or a system retains its intended behavior or output, despite the presence of small errors or deviations in its initial state. This approach not only highlights the robustness of the equation or system but also underscores the importance of stability in the practical application of mathematical theories. In essence, the statement that stability-related outcomes are established by using the Ulam-Hyers (U-H) idea encapsulates a vital principle in the mathematical sciences. It reflects an ongoing effort to ensure that mathematical models not only approximate reality with high fidelity but also maintain their coherence and predictability when faced with inevitable uncertainties. This has far-reaching implications, paving the way for advancements in numerous scientific fields and contributing to the development of technologies that rely on the precise application of mathematical principles. Keeping this importance in mind, we study some results for (U-H) and generalized (U-H) stability for our considered problem. A pertinent example is given to illustrate the results.

2 Basic Results

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Definition 2.1. *[17] Let* u *be the absolutely continuous function on* $\mathcal{H}(0,T)$*, then the ABC derivative for* $\imath_i \in (0,1]$ *is defined as*

$$^{ABC}\mathbb{D}^{\imath_1}\mathrm{u}(\varkappa) = \frac{M(\imath_1)}{1-\imath_1}\int_0^{\varkappa} \mathrm{u}'(s)\mathscr{E}_{\imath_i}\left[\frac{-\imath_i(\varkappa-s)}{1-\imath_i}\right]ds,$$

such that $M(\delta_i)(0) = M(\delta_i)(1) = 1$ *satisfying.*

Definition 2.2. *[17] Let* u *be absolutely integrable function on* $\mathscr{W}$*, then the integral with non-singular kernel for order* $\imath_i \in (0,1]$ *is defined as*

$$^{AB}_{0}\mathbb{I}^{\imath_i}_{\varkappa}\mathrm{u}(\varkappa) = \frac{1-\imath_i}{M(\imath_i)}\mathrm{u}(\varkappa) + \frac{\imath_i}{\Gamma(\imath_i)M(\imath_i)}\int_0^{\varkappa}\mathrm{u}(s)ds.$$

Lemma 2.3. *[17]Let* h *be absolutely integrable function on* $\mathscr{W}$*, and if* $\mathrm{h} \to 0$ *at* $\varkappa = 0$*, then*

$$^{ABC}_{0}\mathbb{D}^{\imath_i}_{\varkappa}\mathrm{u}(\varkappa) = \mathrm{h}(\varkappa),\ 0 < \imath_i \leq 1,$$

has a unique solution described by

$$\mathrm{u}(\varkappa) = \mathrm{u}(0) + \frac{1-\imath_i}{M(\imath_i)}\mathrm{h}(\varkappa) + \frac{\imath_i}{M(\imath_i)\Gamma(\imath_i)}\int_0^{\varkappa}\mathrm{u}(s)ds.$$

Let $\mathrm{X} = C(\mathscr{W}) \times C(\mathscr{W})$ be a Banach space with norm $\|(\mathrm{u},\mathrm{v})\| = \|\mathrm{u}\| + \|\mathrm{v}\|$, where $\|\mathrm{u}\| = \max_{\varkappa\in\mathscr{W}} |\mathrm{u}(\varkappa)|$.

Theorem 2.4. *[31] Let* X *be a Banach space with a closed, convex and bounded subset* $\mathscr{S}$*. Then for two operators* G_1 *and* G_2*, such that*

(i) G_1 *is contraction.*

(ii) G_2 *is completely continuous.*

So, there exist $(\mathrm{u},\mathrm{v}) \in \mathscr{S}$ *such that* $\mathrm{G}_1(\mathrm{u},\mathrm{v}) + \mathrm{G}_2(\mathrm{u},\mathrm{v}) = (\mathrm{u},\mathrm{v})$*, has at least one solution.*

3 Existence Theory

Here we enrich this part by our first main result.

Lemma 3.1. *Let the right sides vanish at* $\varkappa \to 0$ *and if* h_1, h_2 *are absolutely integrable on* $\mathscr{W}$*, then the solution of*

$$\begin{cases} ^{ABC}\mathbb{D}^{\imath_1}\mathrm{u}(\varkappa) = h_1(\varkappa),\ \ 0 < \imath_1 \leq 1, \\ ^{ABC}\mathbb{D}^{\imath_2}\mathrm{v}(\varkappa) = h_2(\varkappa),\ \ 0 < \imath_2 \leq 1, \\ \mathrm{u}(0) = \int_0^T g_1(\mathrm{v}(s))ds, \\ \mathrm{v}(0) = \int_0^T g_2(\mathrm{u}(s))ds, \end{cases} \tag{3.1}$$

is given by

$$\begin{cases} \mathrm{u}(\varkappa) = \int_0^T g_1(\mathrm{v}(s))ds + \frac{(1-\imath_1)h_1(\varkappa)}{M(\imath_1)} + \frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)}\int_0^{\varkappa}(\varkappa-s)^{\imath_1-1}h_1(s)ds, \\ \mathrm{v}(\varkappa) = \int_0^T g_2(\mathrm{u}(s))ds + \frac{(1-\imath_2)h_2(\varkappa)}{M(\imath_2)} + \frac{\imath_2}{M(\imath_2)\Gamma(\imath_2)}\int_0^{\varkappa}(\varkappa-s)^{\imath_2-1}h_2(s)ds. \end{cases} \tag{3.2}$$

Proof: Let c_0, d_0 be constants and applying ${}^{AB}\mathbb{I}^{\imath_1}$, ${}^{AB}\mathbb{I}^{\imath_2}$ on (3.1), we have

$$\begin{cases} \mathrm{u}(\varkappa) = c_0 + \dfrac{(1-\imath_1)h_1(\varkappa)}{M(\imath_1)} + \dfrac{\imath_1}{M(\imath_1)\Gamma(\imath_1)} \displaystyle\int_0^{\varkappa} (\varkappa - s)^{\imath_1 - 1} h_1(s)ds, \\ \mathrm{v}(\varkappa) = d_0 + \dfrac{(1-\imath_2)h_2(\varkappa)}{M(\imath_2)} + \dfrac{\imath_2}{M(\imath_2)\Gamma(\imath_2)} \displaystyle\int_0^{\varkappa} (\varkappa - s)^{\imath_2 - 1} h_2(s)ds. \end{cases} \tag{3.3}$$

Using the boundary conditions from (3.1), we have $c_0 = \int_0^T g_1(\mathrm{v}(s))ds$ and $d_0 = \int_0^T g_2(\mathrm{u}(s))ds$. Hence the system (3.3) becomes

$$\begin{cases} \mathrm{u}(\varkappa) = \displaystyle\int_0^T g_1(\mathrm{v}(s))ds + \dfrac{(1-\imath_1)h_1(\varkappa)}{M(\imath_1)} + \dfrac{\imath_1}{M(\imath_1)\Gamma(\imath_1)} \displaystyle\int_0^{\varkappa} (\varkappa - s)^{\imath_1 - 1} h_1(s)ds, \\ \mathrm{v}(\varkappa) = \displaystyle\int_0^T g_2(\mathrm{u}(s))ds + \dfrac{(1-\imath_2)h_2(\varkappa)}{M(\imath_2)} + \dfrac{\imath_2}{M(\imath_2)\Gamma(\imath_2)} \displaystyle\int_0^{\varkappa} (\varkappa - s)^{\imath_2 - 1} h_2(s)ds. \end{cases}$$

Thus, we get the desired solution.

Theorem 3.2. *Inview of Lemma 3.1, the solution of (1.1) is given by*

$$\begin{cases} \mathrm{u}(\varkappa) = \displaystyle\int_0^T g_1(\mathrm{v}(s))ds + \dfrac{(1-\imath_1)f_1(\varkappa,\mathrm{u}(\varkappa),\mathrm{v}(\varkappa))}{M(\imath_1)} + \dfrac{\imath_1}{M(\imath_1)\Gamma(\imath_1)} \displaystyle\int_0^{\varkappa} (\varkappa - s)^{\imath_1 - 1} f_1(s,\mathrm{u}(s),\mathrm{v}(s))ds, \\ \mathrm{v}(\varkappa) = \displaystyle\int_0^T g_2(\mathrm{u}(s))ds + \dfrac{(1-\imath_2)f_2(\varkappa,\mathrm{u}(\varkappa),\mathrm{v}(\varkappa))}{M(\imath_2)} + \dfrac{\imath_2}{M(\imath_2)\Gamma(\imath_2)} \displaystyle\int_0^{\varkappa} (\varkappa - s)^{\imath_2 - 1} f_2(s,\mathrm{u}(s),\mathrm{v}(s))ds. \end{cases} \tag{3.4}$$

Let define two operators $T_1,\ T_2 : \mathrm{X} \to \mathrm{X}$ by

$$\begin{cases} T_1(\mathrm{u},\mathrm{v}) = \displaystyle\int_0^T g_1(\mathrm{v}(s))ds + \dfrac{(1-\imath_1)f_1(\varkappa,\mathrm{u}(\varkappa),\mathrm{v}(\varkappa))}{M(\imath_1)} + \dfrac{\imath_1}{M(\imath_1)\Gamma(\imath_1)} \displaystyle\int_0^{\varkappa} (\varkappa - s)^{\imath_1 - 1} f_1(s,\mathrm{u}(s),\mathrm{v}(s))ds, \\ T_2(\mathrm{u},\mathrm{v}) = \displaystyle\int_0^T g_2(\mathrm{u}(s))ds + \dfrac{(1-\imath_2)f_2(\varkappa,\mathrm{u}(\varkappa),\mathrm{v}(\varkappa))}{M(\imath_2)} + \dfrac{\imath_2}{M(\imath_2)\Gamma(\imath_2)} \displaystyle\int_0^{\varkappa} (\varkappa - s)^{\imath_2 - 1} f_2(s,\mathrm{u}(s),\mathrm{v}(s))ds \end{cases} \tag{3.5}$$

and $\mathbb{T} = (T_1,\ T_2)(\mathrm{u},\mathrm{v})$.
The following hypothesis hold:

(H_1) For constants $C_{g_1},\ C_{g_2} > 0$ and $\mathrm{u},\ \mathrm{v}, \bar{\mathrm{u}}, \bar{\mathrm{v}} \in \mathrm{X}$ one has

$$\begin{aligned} |g_1(\mathrm{v}) - g_1(\bar{\mathrm{v}})| &\leq C_{g_1}|\mathrm{v} - \bar{\mathrm{v}}| \\ \text{and } |g_2(\mathrm{u}) - g_2(\bar{\mathrm{u}})| &\leq C_{g_2}|\mathrm{u} - \bar{\mathrm{u}}|. \end{aligned}$$

(H_2) For constants $L_{f_1},\ L_{f_2} > 0$, $(\mathrm{u},\mathrm{v})\ (\bar{\mathrm{u}},\bar{\mathrm{v}}) \in \mathrm{X}$, we have

$$|f_i(\varkappa,\mathrm{u},\mathrm{v}) - f_i(\varkappa,\bar{\mathrm{u}},\bar{\mathrm{v}})| \leq L_{f_i}[|\mathrm{u} - \bar{\mathrm{u}}| + |\mathrm{v} - \bar{\mathrm{v}}|],\ i = 1,2.$$

(H_3) For constants $M_{f_1},\ M_{f_2} > 0,\ M_{g_1},\ M_{g_2} > 0$, one has

$$\begin{gathered} M_{f_1} = \max_{\varkappa \in \mathscr{W}} |f_1(\varkappa,0,0)|,\ M_{f_2} = \max_{\varkappa \in \mathscr{W}} |f_2(\varkappa,0,0)|, \\ M_{g_1} = \max_{\varkappa \in \mathscr{W}} |g_1(0)| \text{ and } M_{g_2} = \max_{\varkappa \in \mathscr{W}} |g_2(0)|. \end{gathered}$$

For simplicity, we use

$$\Omega = C_{g_1}T + C_{g_2}T + \left(\frac{1}{M(\imath_1)} + \frac{T^{\imath_1}}{M(\imath_1)\Gamma(\imath_1)}\right)L_{f_1} + \left(\frac{1}{M(\imath_2)} + \frac{T^{\imath_1}}{M(\imath_2)\Gamma(\imath_2)}\right)L_{f_2}. \tag{3.6}$$

Theorem 3.3. *Using hypothesis* $(\mathrm{H}_1,\ \mathrm{H}_2)$ *and if* $\Omega < 1$, *as defined in* (3.6), *then the BVP* (1.1) *has a unique solution.*

Proof: Consider $(\mathrm{u},\mathrm{v}),(\bar{\mathrm{u}},\bar{\mathrm{v}}) \in \mathrm{X}$, one has from system (3.5)

$$\begin{aligned}\|T_1(\mathrm{u},\mathrm{v}) - T_1(\bar{\mathrm{u}},\bar{\mathrm{v}})\| &= \max_{\varkappa\in\mathscr{W}} |T_1(\mathrm{u},\mathrm{v})(\varkappa) - T_1(\bar{\mathrm{u}},\bar{\mathrm{v}})(\varkappa)| \\ &\leq \max_{\varkappa\in\mathscr{W}} \Bigg[\int_0^T |g_1(\mathrm{v}(s)) - g_1(\bar{\mathrm{v}}(s))|ds + \frac{(1-\imath_1)}{M(\imath_1)}|f_1(\varkappa,\mathrm{u}(\varkappa),\mathrm{v}(\varkappa)) - f_1(\varkappa,\bar{\mathrm{u}}(\varkappa),\bar{\mathrm{v}}(\varkappa))| \\ &\quad + \frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)}\int_0^\varkappa (\varkappa-s)^{\imath_1-1}|f_1(s,\mathrm{u},\mathrm{v}(s)) - f_1(s,\bar{\mathrm{u}}(s),\bar{\mathrm{v}}(s))|ds\Bigg] \\ &\leq C_{g_1}T||\mathrm{v}-\bar{\mathrm{v}}|| + \frac{(1-\imath_1)}{M(\imath_1)}L_{f_1}[\|\mathrm{u}-\bar{\mathrm{u}}\| + \|\mathrm{v}-\bar{\mathrm{v}}\|] + \frac{T^{\imath_1}}{M(\imath_1)\Gamma(\imath_1)}L_{f_1}[\|\mathrm{u}-\bar{\mathrm{u}}\| + \|\mathrm{v}-\bar{\mathrm{v}}\|].\end{aligned}$$

Hence

$$\|T_1(\mathrm{u},\mathrm{v}) - T_1(\bar{\mathrm{u}},\bar{\mathrm{v}})\| \leq \left[C_{g_1}T + \left(\frac{1}{M(\imath_1)} + \frac{T^{\imath_1}}{M(\imath_1)\Gamma(\imath_1)}\right)L_{f_1}\right][\|\mathrm{u}-\bar{\mathrm{u}}\| + \|\mathrm{v}-\bar{\mathrm{v}}\|]. \tag{3.7}$$

In the same way, we also deduce that

$$\|T_2(\mathrm{u},\mathrm{v}) - T_2(\bar{\mathrm{u}},\bar{\mathrm{v}})| \leq \left[C_{g_2}T^{\imath_1} + \left(\frac{1}{M(\imath_2)} + \frac{T^{\imath_2}}{M(\imath_2)\Gamma(\imath_2)}\right)L_{f_2}\right][\|\mathrm{u}-\bar{\mathrm{u}}\| + \|\mathrm{v}-\bar{\mathrm{v}}\|]. \tag{3.8}$$

From equation (3.7) and (3.8), one has

$$\|\mathbb{T}(\mathrm{u},\mathrm{v}) - \mathbb{T}(\bar{\mathrm{u}},\bar{\mathrm{v}})\| \leq \Omega\|(\mathrm{u},\mathrm{v}) - (\bar{\mathrm{u}},\bar{\mathrm{v}})\|,$$

where Ω is defined in (3.6). Thus, by Banach contraction theorem, BVP (1.1) has a unique solution.

Theorem 3.4. *Under the hypothesis* $(\mathrm{H}_1 - \mathrm{H}_3)$ *and if*

$$\frac{L_{f_1}}{M(\imath_1)} + \frac{L_{f_2}}{M(\imath_2)} < 1,$$

then the BVP (1.1) *has at least one solution.*

Proof: Let us define $\mathrm{G}_1 = (\mathrm{G}_{11},\mathrm{G}_{12})$ as

$$\begin{aligned}\mathrm{G}_{11}\mathrm{u}(\varkappa) &= \frac{(1-\imath_1)}{M(\imath_1)}f_1(\varkappa,\mathrm{u}(\varkappa),\mathrm{v}(\varkappa)), \\ \mathrm{G}_{12}\mathrm{v}(\varkappa) &= \frac{(1-\imath_2)}{M(\imath_2)}f_2(\varkappa,\mathrm{u}(\varkappa),\mathrm{v}(\varkappa)),\end{aligned}$$

then inview of $(\mathrm{H}_1,\mathrm{H}_2)$, one see that for $(\mathrm{u},\mathrm{v}),\ (\bar{\mathrm{u}},\bar{\mathrm{v}}) \in \mathrm{X}$,

$$\|\mathrm{G}_{11}(\mathrm{u},\mathrm{v}) - \mathrm{G}_{11}(\bar{\mathrm{u}},\mathrm{v})\| \leq \frac{L_{f_1}}{M(\imath_1)}\|(\mathrm{u},\mathrm{v}) - (\bar{\mathrm{u}},\bar{\mathrm{v}})\| \tag{3.9}$$

and

$$\|\mathrm{G}_{12}(\mathrm{u},\mathrm{v}) - \mathrm{G}_{12}(\bar{\mathrm{u}},\mathrm{v})\| \leq \frac{L_{f_2}}{M(\imath_1)}\|(\mathrm{u},\mathrm{v}) - (\bar{\mathrm{u}},\bar{\mathrm{v}})\| \tag{3.10}$$

From equations (3.9) and (3.10), one has

$$\|\mathrm{G}_1(\mathrm{u},\mathrm{v})-\mathrm{G}_1(\bar{\mathrm{u}},\bar{\mathrm{v}})\| \leq \left[\frac{L_{f_1}}{M(\imath_1)}+\frac{L_{f_1}}{M(\imath_2)}\right]\|(\mathrm{u},\mathrm{v})-(\bar{\mathrm{u}},\bar{\mathrm{v}})\|.$$

Thus G_1 is a contraction. Now let $\mathscr{S}=\{(\mathrm{u},\mathrm{v})\in \mathrm{X}:\|(\mathrm{u},\mathrm{v})\|\leq\rho\}$ and define $\mathrm{G}_2:\mathscr{S}\to\mathscr{S}$ by $\mathrm{G}_2=(\mathrm{G}_{21},\mathrm{G}_{22})$, such that

$$\begin{cases}\mathrm{G}_{21}(\mathrm{u},\mathrm{v})(\varkappa)=\displaystyle\int_0^T \mathrm{G}_1(\mathrm{v}(s))ds+\frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)}\int_0^{\varkappa}(\varkappa-s)^{\imath_1-1}f_1(s,\mathrm{u}(s),\mathrm{v}(s))ds,\\ \mathrm{G}_{22}(\mathrm{u},\mathrm{v})(\varkappa)=\displaystyle\int_0^T g_2(\mathrm{u}(s))ds+\frac{\imath_2}{M(\imath_2)\Gamma(\imath_2)}\int_0^{\varkappa}(\varkappa-s)^{\imath_2-1}f_2(s,\mathrm{u}(s),\mathrm{v}(s))ds.\end{cases}$$

Now for any $(\mathrm{u},\mathrm{v})\in\mathscr{S}$, one has

$$\begin{aligned}\|\mathrm{G}_{21}(\mathrm{u},\mathrm{v})\| &\leq \max_{t\in\mathscr{W}}\left[\int_0^T|g_1(\mathrm{v}(s))|ds+\frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)}\int_0^{\varkappa}(\varkappa-s)^{\imath_1-1}|f_1(s,\mathrm{u}(s),\mathrm{v}(s))|ds\right]\\ &\leq \max_{t\in\mathscr{W}}\left[\int_0^T|g_1(\mathrm{v}(s))-g_1(0)|+|g_1(0)|ds\right.\\ &\left.+\frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)}\int_0^{\varkappa}(\varkappa-s)^{\imath_1-1}|f_1(s,\mathrm{u}(s),\mathrm{v}(s))-f_1(s,0,0)|+|f_1(s,0,0)|ds\right]\\ &\leq (C_{g_1}\rho+M_{g_1})T+\frac{T^{\imath_1}}{M(\imath_1)\Gamma(\imath_1)}[L_{f_1}\rho+M_{f_1}]\leq\frac{\rho}{2}.\end{aligned}$$

Hence

$$\|\mathrm{G}_{21}(\mathrm{u},\mathrm{v})\|\leq\frac{\rho}{2}. \tag{3.11}$$

In the same way, we have

$$\|\mathrm{G}_{22}(\mathrm{u},\mathrm{v})\|\leq\frac{\rho}{2}. \tag{3.12}$$

From equations (3.11) and (3.12), one has

$$\rho\geq\max\left\{\frac{M_{g_1}T+\frac{T^{\imath_1}M_{f_1}}{M(\imath_1)\Gamma(\imath_1)}}{\frac{1}{2}-C_{g_1}T-\frac{T^{\imath_1}L_{f_1}}{M(\imath_1)\Gamma(\imath_1)}},\frac{M_{g_2}T+\frac{T^{\imath_2}M_{f_2}}{M(\imath_2)\Gamma(\imath_2)}}{\frac{1}{2}-C_{g_2}T-\frac{T^{\imath_2}L_{f_2}}{M(\imath_1)\Gamma(\imath_1)}}\right\}.$$

Hence from (3.11) and (3.12), one has

$$\|\mathrm{G}_2(\mathrm{u},\mathrm{v})\|\leq\rho.$$

Thus G_2 is bounded. Also g_1,g_2,f_1,f_2 are continuous, so G_2 is continuous. Now to show equi-continuity, let $\varkappa_1<\varkappa_2\in\mathscr{W}$, then

$$\begin{aligned}|\mathrm{G}_{21}(\mathrm{u},\mathrm{v})(\varkappa_2)-\mathrm{G}_{21}(\mathrm{u},\mathrm{v})(\varkappa_1)| &\leq \frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)}\left[\int_0^{\varkappa_1}[(\varkappa_1-s)^{\imath_1-1}-(\varkappa_2-s)^{\imath_1-1}]|f_1(s,\mathrm{u}(s),\mathrm{v}(s))|ds\right.\\ &+\left.\int_{\varkappa_1}^{\varkappa_2}(\varkappa_2-s)^{\imath_1-1}|f_1(s,\mathrm{u}(s),\mathrm{v}(s))|ds\right]\\ &\leq \frac{1}{M(\imath_1)\Gamma(\imath_1)}\left[(\varkappa_1^{\imath_1}-\varkappa_2^{\imath_1}+(\varkappa_2-\varkappa_1)^{\imath_1})(L_{f_1}\rho+M_{f_1})\right.\\ &+\left.(\varkappa_2-\varkappa_1)^{\imath_1}(L_{f_1}\rho+M_{f_1})\right],\end{aligned}$$

as right side tends to zero at $\varkappa_1 \to \varkappa_2$, so

$$|\mathrm{G}_{21}\mathrm{u}(\varkappa_2) - \mathrm{G}_{21}\mathrm{u}(\varkappa_1)| \to 0 \ \text{ as } \ \varkappa_1 \to \varkappa_2.$$

In the same way, one has

$$|\mathrm{G}_{22}\mathrm{u}(\varkappa_2) - \mathrm{G}_{22}\mathrm{u}(\varkappa_1)| \to 0 \ \text{ as } \ \varkappa_1 \to \varkappa_2.$$

Therefore,

$$|\mathrm{G}_{2}\mathrm{u}(\varkappa_2) - \mathrm{G}_{2}\mathrm{u}(\varkappa_1)| \to 0 \ \text{ as } \ \varkappa_1 \to \varkappa_2.$$

As G_2 is bounded and continuous, therefore G_2 is uniformly continuous. So, $\mathrm{G}_2 = (\mathrm{G}_{21}, \mathrm{G}_{22})$ is also completely continuous. Hence using Theorem 2.4, the BVP (1.1) has at least one solution.

4 U-H Stability

Since in operator form the proposed system solution in Theorem 3.2 is written as

$$\begin{cases} T_1(\mathrm{u},\mathrm{v})(\varkappa) = (\mathrm{u},\mathrm{v}), \\ T_2(\mathrm{u},\mathrm{v}) = (\mathrm{u},\mathrm{v}), \end{cases} \tag{4.1}$$

Let for any $\epsilon > 0$, we have ϕ independents of u, v such that $\phi : \mathscr{W} \to R$, such that

$$|\phi(\varkappa)| \le \epsilon, \ \ \varkappa \in \mathscr{W}.$$

In addition, it should be noted that $\phi \to 0$ as $\varkappa \to 0$. Then from Theorem 3.2 and in view of (4.1), the solution of

$$\begin{cases} {}^{ABC}\mathbb{D}^{\imath_1}\mathrm{u}(\varkappa) = f_1(\varkappa, \mathrm{u}(\varkappa), \mathrm{v}(\varkappa)) + \phi(\varkappa), \\ {}^{ABC}\mathbb{D}^{\imath_1}\mathrm{v}(\varkappa) = f_2(\varkappa, \mathrm{u}(\varkappa), \mathrm{v}(\varkappa)) + \phi(\varkappa), \\ \mathrm{u}(0) = \int_0^T g_1(\mathrm{v}(s))ds, \ \ \mathrm{v}(0) = \int_0^T g_2(\mathrm{u}(s))ds, \end{cases} \tag{4.2}$$

is given by

$$\begin{aligned} \mathrm{u}(\varkappa) &= \int_0^T g_1(\mathrm{v}(s))ds + \frac{1-\imath_1}{M(\imath_1)} f_1(\varkappa, \mathrm{u}(\varkappa), \mathrm{v}(\varkappa)) + \frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)} \int_0^{\varkappa} (\varkappa - s)^{\imath_1 - 1} f_1(s, \mathrm{u}(s), \mathrm{v}(s))ds \\ &+ \frac{1-\imath_1}{M(\imath_1)} \phi(\varkappa) + \frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)} \int_0^{\varkappa} (\varkappa - s)^{\imath_1 - 1} \phi(s)ds, \\ \mathrm{v}(\varkappa) &= \int_0^T g_2(\mathrm{u}(s))ds + \frac{1-\imath_2}{M(\imath_2)} f_2(\varkappa, \mathrm{u}(\varkappa), \mathrm{u}(\varkappa)) + \frac{\imath_2}{M(\imath_2)\Gamma(\imath_2)} \int_0^{\varkappa} (\varkappa - s)^{\imath_2 - 1} f_2(s, \mathrm{u}(s), \mathrm{v}(s))ds \\ &+ \frac{1-\imath_2}{M(\imath_2)} \phi(\varkappa) + \frac{\imath_2}{M(\imath_2)\Gamma(\imath_2)} \int_0^{\varkappa} (\varkappa - s)^{\imath_2 - 1} \phi(s)ds. \end{aligned}$$

Further, we have

$$\begin{aligned} &\left| \mathrm{v}(\varkappa) - \left(\int_0^T g_1(\mathrm{v}(s))ds + \frac{1-\imath_1}{M(\imath_1)} f_1(\varkappa, \mathrm{u}(\varkappa)\mathrm{v}(\varkappa)) + \frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)} \int_0^{\varkappa} (\varkappa - s)^{\imath_1 - 1} f_1(s, \mathrm{u}(s)\mathrm{v}(s))ds \right) \right| \\ &\le \left(\frac{1}{M(\imath_1)} + \frac{T^{\imath_1}}{M(\imath_1)\Gamma(\imath_1)} \right) \epsilon := A\epsilon, \end{aligned} \tag{4.3}$$

$$\begin{aligned} &\left| \mathrm{v}(\varkappa) - \left(\int_0^T g_2(\mathrm{u}(s))ds + \frac{1-\imath_2}{M(\imath_2)} f_2(\varkappa, \mathrm{u}(\varkappa), \mathrm{u}(\varkappa)) + \frac{\imath_2}{M(\imath_2)\Gamma(\imath_2)} \int_0^{\varkappa} (\varkappa - s)^{\imath_2 - 1} f_2(s, \mathrm{u}(s), \mathrm{v}(s))ds \right) \right| \\ &\le \left(\frac{1}{M(\imath_2)} + \frac{T^{\imath_2}}{M(\imath_2)\Gamma(\imath_2)} \right) \epsilon := B\epsilon. \end{aligned} \tag{4.4}$$

Theorem 4.1. *In view of* (4.3) *and* (4.4) *the solution of BVP* (1.1) *is U-H stable if the condition* $\overbrace{\Delta} < 1$, *where* $\overbrace{\Delta} = \Delta_1 + \Delta_2 + C_{g_1}T\Delta_2 + C_{g_2}T\Delta_1 + C_{g_1}C_{g_2}T^2$ *such that*

$$\Delta_1 = \frac{L_{f_1}T^{\imath_1}}{M(\imath_1)\Gamma(\imath_1)} + \frac{L_{f_1}}{M(\imath_1)}, \tag{4.5}$$

and

$$\Delta_2 = \frac{L_{f_2}T^{\imath_2}}{M(\imath_2)\Gamma(\imath_2)} + \frac{L_{f_2}}{M(\imath_2)}. \tag{4.6}$$

Proof: Let $(\mathrm{u},\mathrm{v}) \in X$ be any solution and $(\bar{\mathrm{u}},\bar{\mathrm{v}}) \in X$ be a unique solution, then

$$\begin{aligned}
|\mathrm{u}-\bar{\mathrm{u}}| &= \left|\mathrm{v}(\varkappa) - \left(\int_0^T g_1(\bar{\mathrm{v}}(s))ds + \frac{1-\imath_1}{M(\imath_1)}f_1(\varkappa,\bar{\mathrm{u}}(\varkappa),\bar{\mathrm{v}}(\varkappa)) + \frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)}\int_0^{\varkappa}(\varkappa-s)^{\imath_1-1}f_1(s,\bar{\mathrm{u}}(s)\bar{\mathrm{v}}(s))ds\right)\right| \\
&\leq \left|\mathrm{u}(\varkappa) - \left(\int_0^T g_1(\bar{\mathrm{v}}(s))ds + \frac{1-\imath_1}{M(\imath_1)}f_1(\varkappa,\bar{\mathrm{u}}(\varkappa)\bar{\mathrm{v}}(\varkappa)) + \frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)}\int_0^{\varkappa}(\varkappa-s)^{\imath_1-1}f_1(s,\bar{\mathrm{u}}(s),\bar{\mathrm{v}}(s))ds\right)\right| \\
&+ \int_0^T |g_1(\mathrm{v}(s)) - g_1(\bar{\mathrm{v}}(s))|ds + \frac{1-\imath_1}{M(\imath_1)}\left|f_1(\varkappa,\mathrm{u}(\varkappa),\mathrm{v}(\varkappa)) - f_1(\varkappa,\bar{\mathrm{u}}(\varkappa),\bar{\mathrm{v}}(\varkappa))\right| \\
&+ \frac{\imath_1}{M(\imath_1)\Gamma(\imath_1)}\int_0^{\varkappa}(\varkappa-s)^{\imath_1-1}\left|f_1(s,\mathrm{u}(s)\mathrm{v}(s)) - f_1(s,\bar{\mathrm{u}}(s),\bar{\mathrm{v}}(s))\right|ds \\
&\leq A\epsilon + C_{g_1}T\|\mathrm{v}-\bar{\mathrm{v}}\| + \left[\frac{L_{f_1}}{M(\imath_1)} + \frac{L_{f_1}T^{\imath_1}}{M(\imath_1)\Gamma(\imath_1)}\right]\|(\mathrm{u},\mathrm{v}) - (\bar{\mathrm{u}},\bar{\mathrm{v}})\|.
\end{aligned} \tag{4.7}$$

After some further simplification of (4.7), and using (4.5), we have

$$\|\mathrm{u}-\bar{\mathrm{u}}\| \leq A\epsilon + \Delta_1\|\mathrm{u}-\bar{\mathrm{u}}\| + (C_{g_1}T + \Delta_1)\|\mathrm{v}-\bar{\mathrm{v}}\|. \tag{4.8}$$

In the same way, by using (4.6), one can also get

$$\|\mathrm{v}-\bar{\mathrm{v}}\| \leq B\epsilon + (C_{g_2}T + \Delta_2)\|\mathrm{u}-\bar{\mathrm{u}}\| + \Delta_2\|\mathrm{v}-\bar{\mathrm{v}}\|. \tag{4.9}$$

Then, from (4.8), and (4.9), one has

$$\begin{aligned}
(1-\Delta_1)\|\mathrm{u}-\bar{\mathrm{u}}\| - (C_{g_1}T + \Delta_1)\|\mathrm{v}-\bar{\mathrm{v}}\| &\leq A\epsilon \\
-(C_{g_2}T + \Delta_1)\|\mathrm{u}-\bar{\mathrm{u}}\| + (1-\Delta_2)\|\mathrm{v}-\bar{\mathrm{v}}\| &\leq B\epsilon,
\end{aligned} \tag{4.10}$$

which can be expressed as

$$\begin{bmatrix} (1-\Delta_1) & -(C_{g_1}T+\Delta_1) \\ -(C_{g_2}T+\Delta_2) & (1-\Delta_2) \end{bmatrix} \begin{bmatrix} \|\mathrm{u}-\bar{\mathrm{u}}\| \\ \|\mathrm{v}-\bar{\mathrm{v}}\| \end{bmatrix} \leq \begin{bmatrix} A \\ B \end{bmatrix} \epsilon.$$

On solving the above inequity, we have by using

$$\|\mathrm{u}-\bar{\mathrm{u}}\| \leq \left(\frac{A(1-\Delta_2) + B(C_{g_1}T + \Delta_1)}{1-\overbrace{\Delta}}\right)\epsilon$$

and

$$\|\mathrm{v}-\bar{\mathrm{v}}\| \leq \left(\frac{B(1-\Delta_1) + A(C_{g_2}T + \Delta_2)}{1-\overbrace{\Delta}}\right)\epsilon$$

which further yields that by using

$$\max\left\{A(1-\Delta_2)+B(C_{g_1}T+\Delta_1),\ B(1-\Delta_1)+A(C_{g_2}T+\Delta_2)\right\}=\Theta_{\Delta_1,\Delta_2,T,C_{g_1},C_{g_2}},$$

$$\|(\mathrm{u},v)-(\bar{\mathrm{u}},\bar{\mathrm{v}})\|\leq\frac{\Theta_{\Delta_1,\Delta_2,T,C_{g_1},C_{g_2}}}{1-\overbrace{\Delta}}\epsilon. \tag{4.11}$$

Thus the solution is U-H stable.

In addition if there exists a non-decreasing function $\psi:[0,T]\to R$, with $\psi(\epsilon)=\epsilon$ and $\psi(0)=0$. Then from (4.11), we have

$$\|(\mathrm{u},\mathrm{v})-(\bar{\mathrm{u}},\bar{\mathrm{v}})\|\leq\frac{\Theta_{\Delta_1,\Delta_2,T,C_{g_1},C_{g_2}}}{1-\overbrace{\Delta}}\psi(\epsilon).$$

Hence the solution is generalized U-H stable.

5 Application

Here, we present an example as illustration of our main results.

Example 5.1. *Consider*

$$\begin{cases}{}^{ABC}\mathbb{D}^{\frac{1}{2}}\mathrm{u}(\varkappa)=\frac{\exp(-\varkappa)+|\mathrm{u}(\varkappa)|+\sin|\mathrm{v}(\varkappa)|}{\varkappa^3+80},\\ {}^{ABC}\mathbb{D}^{\frac{1}{2}}\mathrm{v}(\varkappa)=\frac{\exp(-4\pi)+|\sin\mathrm{u}(\varkappa)|+|\mathrm{v}(\varkappa)|}{\varkappa^3+80},\\ \mathrm{u}(0)=\int_0^1\frac{\exp(-|\mathrm{v}(s)|)}{s^2+100}ds,\quad \mathrm{v}(0)=\int_0^1\frac{\exp-|\mathrm{u}(s)|}{s^4+100}ds.\end{cases} \tag{5.1}$$

Then

$$f_1(\varkappa,\mathrm{u}(\varkappa),\mathrm{v}(\varkappa))=\frac{\exp(-\varkappa)+|\mathrm{u}(\varkappa)|+\sin|\mathrm{v}(\varkappa)|}{\varkappa^3+80},\ f_2(\varkappa,\mathrm{u}(\varkappa),\mathrm{v}(\varkappa))=\frac{\exp(-4\pi)+|\sin\mathrm{u}(\varkappa)|+|\mathrm{v}(\varkappa)|}{\varkappa^3+80},$$

$$g_1(\mathrm{v}(\varkappa))=\frac{\exp(-|\mathrm{v}(\varkappa)|)}{\varkappa^2+100},\quad g_2(\mathrm{u}(\varkappa))=\frac{\exp(-|\mathrm{u}(\varkappa)|)}{\varkappa^2+100}.$$

On calculation, we have $L_{f_i}=\frac{1}{80}$, $M_{f_i}=0$, *for* $i=1,2$, $C_{g_i}=\frac{1}{100}$, $M_{g_i}=1$, *for* $i=1,2$. *Also using* $M(\frac{1}{2})=1$, $T=1$, *then one can compute that*

$$\begin{aligned}\Omega &= C_{g_1}T+C_{g_2}T+\left(\frac{1}{M(\imath_1)}+\frac{T^{\imath_1}}{M(\imath_1)\Gamma(\imath_1)}\right)L_{f_1}+\left(\frac{1}{M(\imath_2)}+\frac{T^{\imath_1}}{M(\imath_2)\Gamma(\imath_2)}\right)L_{f_2}\\ &= \frac{1}{100}.1+\frac{1}{100}.1+2\left(1+\frac{1}{\Gamma(\frac{1}{2})}\right)\frac{1}{80}\\ &= 0.60918958<1.\end{aligned}$$

Hence inview of Theorem 3.3, the given problem has a unique solution. Moreover, the conditions of Theorem 3.4 are also satisfied and we see that

$$\frac{L_{f_1}}{M(\imath_1)}+\frac{L_{f_2}}{M(\imath_2)}=\frac{2}{80}<1.$$

Hence the given problem has at least one solution. To see condition of U-H stability, we see that $\Delta_i=\frac{1+\sqrt{\pi}}{80\sqrt{\pi}}$, $i=1,2$, *and*

$$\begin{aligned}\overbrace{\Delta} &= \Delta_1+\Delta_2+C_{g_1}T\Delta_2+C_{g_2}T\Delta_1+C_{g_1}C_{g_2}T^2\\ &= \frac{2(1+\sqrt{\pi})}{80\sqrt{\pi}}+\frac{2}{100}\frac{(1+\sqrt{\pi})}{80\sqrt{\pi}}+\frac{1}{10000}<1,\end{aligned}$$

Therefore, inview of Theorem 4.1, the solution is U-H stable.

6 Conclusion

This work has been committed to discuss the existence theory of coupled BVP with coupled integral boundary conditions. Additionally, we have investigated the mentioned problems by using the *ABC* fractional order differential operator. On using the Krasnoselskii and Banach fixed point theorems, we have established sufficient conditions for the existence and uniqueness of the solution. Also, using the tools of numerical functional analysis, a result related to U-H stability has also been derived. By a pertinent example, we have testified the results. For future work, we recommend this work can be extended to the study of numerical analysis. Also another approaches for qualitative study can be applied to the consider problem, such as topological degree theory and Measure of non-compactness.

Author Contributions

All of the work in this article is provided by the single author.

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