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Numerical Analysis of Fractional Order Drinking Mathematical Model

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ABSTRACT

This manuscript is concerned to fractional order SMR-type alcohol drinking model that shows the interaction between alcohol drinkers and non consumers of alcohols. In the model, the whole population is classified into three different classes regarding to their alcohol utilization, namely, Susceptible class S, i.e non consumers, Moderate consumers M, and Risk consumers R. Since, alcohol consumption is a risk factor for various chronic diseases like Psychiatric conditions, cardiovascular diseases, digestive issues, and certain types of cancers. That's why the qualitative and quantitative behavior of the alcohol drinking model is analyzed in this research article. The authors used the results of fixed point theory and the results of Ulam stability to analyze the model qualitatively. For the quantitative analysis, we have constructed a general scheme for solution of proposed model by using the two steps Adam's Bashforth method involving Caputo's fractional derivative. The structure of the method converges to the traditional Adams-Bashforth technique when the fractional order derivatives approach the conventional derivative. The constructed scheme is also authenticated through numerical example. Finally, the results obtained are simulated graphically using Matlab.

Keywords: Fractional Drinking Model; Numerical Approximation; Caputo's Derivative; Fractional Adam's Bshforth Scheme; Fractional Differential Equations; Numerical Simulation.

1 Introduction

In recent decades, the impact of alcohol addiction has gone far beyond the financial expenses; when an individual has an alcohol problem, it can disrupt his marriage and his entire family life. There is also a negative influence on the neighbourhood, schools, workplaces, the health care system, and society as a whole. Alcohol intake is a major risk factor for a wide range of chronic illnesses and disorders. As a result of alcoholism, an individual may face certain types of cancers, Psychiatric conditions, and cardiovascular as well as digestive diseases. It can also increase the risk of diabetes, stoke, and heart diseases. In the last two decades, researchers have tried their level best to trace out the reasons that are accountable for the spread of deadly diseases in society. they have developed certain types of epidemiological models to fix the reasons for such diseases and optimize the spread in the community. [1–5]. By considering the epidemiological models, the researchers have extended their studies to social norms like smoking [8], consumption of alcohol, chubbiness, narcotics, radicalization incidents, etc., (for details see [9, 10]). The principal cause of the aforesaid social norms can be modeled by contagious phenomena. Social or peer stress and decisive reinforcement from another source can control each other's system of life by bringing positive or negative consequences to the individual. Hence the

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researchers also used to describe tendencies like alcohol with the help of the mathematical models for infectious disease [11, 12].

Fractional-order mathematical models are vital and have numerous advantages in representing the dynamical behavior of real-world problems. Further, it has higher accuracy and more reliability than the conventional order derivatives (CoD's) due to the additional properties of hereditary and memory depiction [13]. It is well known that the derivative of classical order or integer order could not scrutinize the dynamical behavior of the problems in between two distinct points. Therefore, the researchers put their efforts forward to introduce the new concept of derivative having fractional or arbitrary order which can easily interpret the dynamical behavior of the problems in between any two distinct points. Numerous researchers have presented their definitions for the depiction of the fractional derivatives at the initial stages, but Riemann and Lioulle were the first two mathematicians who succeeded in giving the then-recognized definition of the arbitrary order derivative named RL fractional derivative [14]. Due to some demerits of the RL derivative, the definition is further modified by a well-known mathematician Michelle Caputo in 1967 in his seminal paper [6]. Based on this new notion, Caputo and Fabrizio put forward a new definition for the fractional order involving exponential kernel termed as Caputo-Fabrizio fractional derivative [15]. Abdon Atangan and Dumitru Baleanu the two famous mathematicians came up with an idea for a new definition for the arbitrary order derivative involving the Mittag-Leffler kernel a non-local and nonsingular kernel named as AB fractional derivative in Caputo's and Riemann Lioulle's sense separately [16, 17].

It is well known that the Adams-Bashforth approach is an excellent and potent numerical integration technique that can yield a numerical solution of ordinary differential equations that is closer to the precise solution. The advantages of this technique include its accuracy, stability, flexibility, ease of implementation, and local error implementation. However, it has the constraints of being sensitive to step size, having a limited stability zone, not being self-starting, and not being suited for dealing with partial differential equations. The scheme of this technique was created by using classical differentiation, which involves Lagrange interpolation and calculation of the difference between two times, such as t_{n+1} and t_n. Later on, The idea of fractional differentiation using Caputo and Riemann-Liouville derivatives was subsequently introduced to this technique. But since fractional integration's kernel is nonlinear, the adaptation was incorrect theoretically. Furthermore, we were unable to retrieve the conventional Adams-Bashforth numerical method using this constructed fractional version when the fractional order = 1. In this article, we present a new scheme of the Adams-Bashforth method for handling fractional differentiation with Caputo derivative, which is predicted to recover the conventional Adams-Bashforth method when the fractional order flips to 1.

In this paper, we have presented the mathematical model for drinking (alcohol, etc) with fractional order Caputo's derivative which has not yet been studied. The present paper has been organized into eight sections. Section 1, of the paper, contains the introduction of the paper. In Section 2, A brief explanation of the model's formulation has been provided while, Section 3, is dedicated to the preliminaries of the research study involved in this paper. Section 4, has been restricted to provide the qualitative analysis of the considered model while in Section 5, the authors have presented the stability analysis. Furthermore, the general scheme for numerical solution of the model is presented in Section 6, and the numerical simulation has been given in Section 7. Finally, the last Section concludes the current research work.

2 Model formulation

In this section, the authors extended the model presented by Nuno et al, in [18], where the population has been considered as homogeneous mixing, by which we mean that the population is connected fully by N individuals. Further the details of the division of the population into *Three* compartments are given in the following.

- **S**: This class of the population consists of individuals that do not consume alcohol. This class has been named as Susceptible **S**.
- **M**: This second class of the model has been devoted to individuals who do consumes the alcohols but at low amount. This class has been given the name of Moderate drinkers **M**.
- **R** : The third and last class of the model contains the individuals of the population who consumes the alcohol at regular basis at high amount. This class has been named as Risk drinkers **R**.

We call the individuals moderate drinkers whose consumption of alcohol is less than $50cc/day$. For women the amount has been fixed to less than $30cc/day$. While the risk drinkers (men and women) are individuals whose consumption of the alcohol per day is more than the moderate drinkers [10,18,19]. The transitions among compartments are as following,

- $\mathbf{S} \xrightarrow{\varphi} \mathbf{M}$: an individual of the population which belongs to the $(\mathbf{S})$ class is converted to $(\mathbf{M})$ class with rate φ if the individual comes in contact with the drinkers of either $(\mathbf{M})$ or $(\mathbf{R})$ class.
- $\mathbf{M} \xrightarrow{\mu} \mathbf{R}$: the individuals (men & women) of $(\mathbf{M})$ class enters into $(\mathbf{R})$ class with the rate μ.
- $\mathbf{M} \xrightarrow{\delta} \mathbf{R}$: the individuals of class M transfer to the class of risk drinker $(\mathbf{R})$ with the rate δ when they come in contact with the individuals of class $(\mathbf{R})$:
- $\mathbf{R} \xrightarrow{\gamma} \mathbf{S}$: The members of $(\mathbf{R})$ join the $(\mathbf{S})$ class with the rate γ if they come in contact with the members of class $(\mathbf{S})$.

The parameter φ is used for the rate of infection, which means that the rate at which a nonconsumer of alcohol starts consuming alcohol. The parameter δ has been used to symbolize the rate of converting the individuals of **M** class to the risk drinkers class **R**. The parameter μ is specified for the rate at which individuals consume alcohol extensively and put impact on others which results the transition $\mathbf{M} \rightarrow \mathbf{R}$. The rate of per-capita new inclusion of the individuals to susceptible class has been symbolized by Λ. While η has been given the role of representing deaths that occur naturally. On behalf of these assumptions, we present the following SMR type model consisting of three ordinary differential equations:

$$\begin{aligned}
\frac{dS(\mathrm{t})}{dt} &= \Lambda - \frac{\varphi S(\mathrm{t})M(\mathrm{t})}{N} - \frac{\varphi S(\mathrm{t})R(\mathrm{t})}{N} + \frac{\delta S(\mathrm{t})R(\mathrm{t})}{N} - \eta S(\mathrm{t}),\\
\frac{dM(\mathrm{t})}{dt} &= \frac{\varphi S(\mathrm{t})M(\mathrm{t})}{N} + \frac{\varphi S(\mathrm{t})R(\mathrm{t})}{N} - \frac{\gamma M(\mathrm{t})R(\mathrm{t})}{N} - (\eta+\mu)M(\mathrm{t}),\\
\frac{dR(\mathrm{t})}{dt} &= \mu M(\mathrm{t}) + \frac{\gamma M(\mathrm{t})R(\mathrm{t})}{N} - \frac{\delta S(\mathrm{t})R(\mathrm{t})}{N} - \eta R(\mathrm{t}).
\end{aligned} \tag{2.1}$$

The fractional form of (2.1) is expressed as

$$\begin{aligned}
{}^{c}D^{p}S(\mathrm{t}) &= \Lambda - \frac{\varphi S(\mathrm{t})M(\mathrm{t})}{N} - \frac{\varphi S(\mathrm{t})R(\mathrm{t})}{N} + \frac{\delta S(\mathrm{t})R(\mathrm{t})}{N} - \eta S(\mathrm{t}),\\
{}^{c}D^{p}M(\mathrm{t}) &= \frac{\varphi S(\mathrm{t})M(\mathrm{t})}{N} + \frac{\varphi S(\mathrm{t})R(\mathrm{t})}{N} - \frac{\gamma M(\mathrm{t})R(\mathrm{t})}{N} - (\eta+\mu)M(\mathrm{t}),\\
{}^{c}D^{p}R(\mathrm{t}) &= \mu M(\mathrm{t}) + \frac{\gamma M(\mathrm{t})R(\mathrm{t})}{N} - \frac{\delta S(\mathrm{t})R(\mathrm{t})}{N} - \eta R(\mathrm{t}).
\end{aligned} \tag{2.2}$$

3 Preliminaries

Definition 3.1. *The fractional Caputo's derivative of order $p > 0$ for a function $S(\mathrm{t})$ is defined as*

$$^{C}D^{p}S(\mathrm{t}) = \frac{1}{\Gamma(p)}\int_0^t (\mathrm{t}-\xi)^{p-1}S^{n}(\xi)d\xi, \quad 0 < p \leq 1.$$

Lemma 3.2. [20] *Let $S(\theta) \in C(0,T)$, then the solution of the FDE*

$$\begin{cases} ^{C}D_0^{p}\mathbb{X}(\theta) = S(\theta), \ \theta \in J = [0,T], \\ \mathbb{X}(0) = S_0 \end{cases}$$

is given by

$$\mathbb{X}(\theta) = \sum_{j=0}^{p} N_j\theta^j + \frac{1}{\Gamma(\mu)}\int_0^\theta (\theta-\xi)^{\mu-1}S(\xi)d\xi. \tag{3.1}$$

For $N_j \in R$, j= 0, 1, 2, 3,..., p.

Definition 3.3. *Consider the FDE having Caputo's fractional derivative of order p*

$$^{c}D^{p}S(\mathrm{t}) = f(\mathrm{t}, S(\mathrm{t})).$$

Then the numerical approximation under FABs [22] is given as

$$\begin{aligned} S(\mathrm{t}_{n+1}) = S(\mathrm{t}_n) &+ \frac{f(\mathrm{t}_n, S_n)}{i\Gamma(p)}\left\{\frac{2i}{p}t_{n+1}^{p} - \frac{t_{n+1}^{p+1}}{p+1} + \frac{i}{p}t_n^{p} - \frac{t_n^{p+1}}{p}\right\} \\ &+ \frac{f(\mathrm{t}_{n-1}, S_{n-1})}{i\Gamma(p)}\left\{\frac{i}{p}t_{n+1}^{p} - \frac{t_{n+1}^{p+1}}{p+1} + \frac{t_n^{p}}{p+1}\right\}. \end{aligned} \tag{3.2}$$

The convergence of the scheme is presented in [22].

Theorem 3.4. *Let $\left\|f^3(t, y(t))\right\|_\infty < Q < \infty$ and*

$$^{c}D^{p}S(\mathrm{t}) = f(\mathrm{t}, S(\mathrm{t})),$$

be a fractional DE such that f is bounded, then $S(\mathrm{t})$ has the numerical solution given by

$$\begin{aligned} S(\mathrm{t}_{n+1}) = S(\mathrm{t}_n) &+ \frac{f(\mathrm{t}_n, S_n)}{i\Gamma(p)}\left\{\frac{2i}{p}t_{n+1}^{p} - \frac{t_{n+1}^{p+1}}{p+1} + \frac{i}{p}t_n^{p} - \frac{t_n^{p+1}}{p}\right\} \\ &+ \frac{f(\mathrm{t}_{n-1}, S_{n-1})}{i\Gamma(p)}\left\{\frac{i}{p}t_{n+1}^{p} - \frac{t_{n+1}^{p+1}}{p+1} + \frac{t_n^{p}}{p+1}\right\} + G_n^{p}(t), \end{aligned} \tag{3.3}$$

where,

$$G_n^{p}(t) < \frac{i^{3+p}Q}{12\Gamma(p+1)}\{(n+1)^p + n^2\}.$$

For a comparable proof, readers are directed to the most current work by Atangana and Owolabi [22].

Theorem 3.5. [21] *"A compact and continuous operator $\wp : \mathscr{Y} \to \mathscr{Y}$ has said to have a unique fixed point if the set defined by*

$$\mathscr{E} = \{\mathbb{X} \in Y : \mathbb{X} = m_1\wp\mathbb{X}, \ m_1 \in (0,1)\},$$

is bounded".

4 Qualitative Analysis

For the existence of solution of the model, we define a function in the following manner

$$\begin{cases} \mathbb{X}_1(\mathrm{t}, S, M, R) = \Lambda - \dfrac{\varphi S(\mathrm{t}) M(\mathrm{t})}{N} - \dfrac{\varphi S(\mathrm{t}) R(\mathrm{t})}{N} + \dfrac{\delta S(\mathrm{t}) R(\mathrm{t})}{N} - \eta S(\mathrm{t}), \\ \mathbb{X}_2(\mathrm{t}, S, M, R) = \dfrac{\varphi S(\mathrm{t}) M(\mathrm{t})}{N} + \dfrac{\varphi S(\mathrm{t}) R(\mathrm{t})}{N} - \dfrac{\gamma M(\mathrm{t}) R(\mathrm{t})}{N} - (\eta + \mu) M(\mathrm{t}), \\ \mathbb{X}_3(\mathrm{t}, S, M, R) = \mu M(\mathrm{t}) + \dfrac{\gamma M(\mathrm{t}) R(\mathrm{t})}{N} - \dfrac{\delta S(\mathrm{t}) R(\mathrm{t})}{N} - \eta R(\mathrm{t}). \end{cases} \tag{4.1}$$

Assume that we have a Banach space $\mathbb{B} = CJ$ in which the norm is defined as

$$||\mathbb{X}(\mathrm{t})|| = \sup_{t \in J} \Big[|S(\mathrm{t})| + |M(\mathrm{t})| + |R(\mathrm{t})| \Big],$$

where

$$\mathbb{X}(\mathrm{t}) = \begin{cases} S(\mathrm{t}) \\ M(\mathrm{t}) \\ R(\mathrm{t}) \end{cases}, \quad \mathbb{X}_0(\mathrm{t}) = \begin{cases} S_0 \\ M_0 \\ R_0 \end{cases}, \quad \Psi(\mathrm{t}, \mathbb{X}(\mathrm{t})) = \begin{cases} \mathbb{X}_1(\mathrm{t}, S, M, R) \\ \mathbb{X}_2(\mathrm{t}, S, M, R) \\ \mathbb{X}_3(\mathrm{t}, S, M, R) \end{cases}. \tag{4.2}$$

By making use of (4.2), the system (2.2) can be represented as

$$\begin{aligned} &{}^{C}D^{p}\mathbb{X}(\mathrm{t}) = \Psi(\mathrm{t}, \mathbb{X}(\mathrm{t})), \quad t \in J, \\ &\mathbb{X}(0) = \mathbb{X}_0, \end{aligned} \tag{4.3}$$

with the help of Lemma (3.2), equation (4.3) becomes

$$\mathbb{X}(\mathrm{t}) = \mathbb{X}_0 + \frac{1}{\Gamma(p)} \int_0^t (\mathrm{t} - \xi)^{p-1} \Psi(\xi, \mathbb{X}(\xi)) d\xi, \ t \in J. \tag{4.4}$$

To prove the existence of solution of the model we presented, we assume that (P_1) and (P_2) are true

$(P1)$ For the constants $\mathfrak{K}^{**}, M_*$, we have

$$|\Psi(\mathrm{t}, \mathbb{X}(\mathrm{t}))| \leq \mathfrak{K}^{**} |\mathbb{X}|^q + M_*.$$

$(P2)$ For $L_* > 0$, and for each $\mathbb{X}, \bar{\mathbb{X}}$, we have

$$|\Psi(\mathrm{t}, \mathbb{X}) - \Psi(\mathrm{t}, \bar{\mathbb{X}})| \leq L_* ||\mathbb{X} - \bar{\mathbb{X}}||.$$

Now considering an operator $\mathtt{T} : \mathbb{B} \to \mathbb{B}$ as

$$\mathtt{T}\mathbb{X}(\mathrm{t}) = \mathbb{X}_0 + \frac{1}{\Gamma(p)} \int_0^t (\mathrm{t} - \xi)^{p-1} \Psi(\xi, \mathbb{X}(\xi)) d\xi. \tag{4.5}$$

Theorem 4.1. *With the usage of the axioms $(P1)$ and $(P2)$, system (2.2) has at least one solution.*

Proof. To obtain our desired result, we need to prove that system (4.3) has at least one fixed point. For this we will make use of Schaefer's fixed point theorem, whose procedure is discussed under.
Step I: In this stage, we will prove the continuity of the operator $\mathtt{T}$. Let $\mathbb{X}_n \to \mathbb{X}$, for $\mathbb{X}_n, \mathbb{X} \in B$.

$$\begin{aligned} ||\mathtt{T}\mathbb{X}_n - \mathtt{T}\mathbb{X}|| &= \max_{t \in J} \left| \frac{1}{\Gamma(p)} \int_0^t (\mathrm{t} - \xi)^{p-1} \Psi_n(\xi, \mathbb{X}_n(\xi)) d\xi - \frac{1}{\Gamma(p)} \int_0^t (\mathrm{t} - \xi)^{p-1} \Psi(\xi, \mathbb{X}(\xi)) d\xi \right| \\ &\leq \max_{t \in J} \int_0^t \left| \frac{(\mathrm{t} - \xi)^{p-1}}{\Gamma(p)} \right| |\Psi_n(\xi, \mathbb{X}_n(\xi)) - \Psi(\xi, \mathbb{X}(\xi))| d\xi \\ &\leq \frac{T^t}{\Gamma(p+1)} ||\Psi_n - \Psi|| \to 0 \ \ as \ \ n. \end{aligned} \tag{4.6}$$

As we know about the continuity of Ψ, therefore $\mathtt{T}\mathbb{X}_n \to \mathtt{T}\mathbb{X}$ which means $\mathtt{T}$ is continuous.
Step II: This stage is related to show that the operator $\mathtt{T}$ is bounded. Suppose, for any $\mathbb{X} \in B$, the operator $\mathtt{T}$ satisfies the growth conditions stated below:

$$\begin{aligned} ||\mathtt{T}\mathbb{X}|| &= \max_{t\in J} \left| \mathbb{X}_0 + \frac{1}{\Gamma(p)} \int_0^t (\mathrm{t}-\xi)^{p-1} \Psi(\xi, \mathbb{X}(\xi)) d\xi \right|, \\ &\leq |\mathbb{X}_0| + \max_{t\in J} \frac{1}{\Gamma(p)} \int_0^t \left| (\mathrm{t}-\xi)^{p-1} \right| |\Psi(\xi, \mathbb{X}(\xi))| d\xi, \\ &\leq |\mathbb{X}_0| + \frac{T^p}{\Gamma(p+1)} [\mathfrak{K}^{**} ||\mathbb{X}||^q + M_*]. \end{aligned} \tag{4.7}$$

Further, let $\mathbb{S}$ be a bounded subset of B. Then $\exists\ K_q \geq 0, \ni$

$$||\mathbb{X}|| \leq K_q, \forall\ \ \mathbb{X} \in \mathbb{S}. \tag{4.8}$$

Now, with the help of the above growth condition, for any $\mathbb{X}_n, \mathbb{X} \in B$, we have

$$\begin{aligned} ||\mathtt{T}\mathbb{X}|| &\leq |\mathbb{X}_0| + \frac{T^p}{\Gamma(p+1)} [\mathfrak{K}^{**} ||\mathbb{X}^q|| + M_*], \\ &\leq |\mathbb{X}_0| + \frac{T^p}{\Gamma(p+1)} [K_* K_q + M_*]. \end{aligned} \tag{4.9}$$

Thus, $\mathtt{T}(\mathbb{S})$ is bounded.
Step III: To prove the equi-continuty, assume $t_1, t_2 \in J$, such that $t_1 \geq t_2$ then

$$\begin{aligned} |\mathtt{T}\mathbb{X}(\mathrm{t}_1) - \mathtt{T}\mathbb{X}(\mathrm{t}_2)| &= \left| \frac{1}{\Gamma(p)} \int_0^t (\mathrm{t}_1-\xi)^{p-1} \Psi(\xi, \mathbb{X}(\xi)) d\xi - \frac{1}{\Gamma(p)} \int_0^t (\mathrm{t}_2-\xi)^{p-1} \Psi(\xi, \mathbb{X}(\xi)) d\xi \right|, \\ &\leq \left| \frac{1}{\Gamma(p)} \int_0^t (\mathrm{t}_1-\xi)^{p-1} - \frac{1}{\Gamma(p)} \int_0^t (\mathrm{t}_2-\xi)^{p-1} \right| |\Psi(\xi, \mathbb{X}(\xi))| d\xi, \\ &\leq \frac{T^p}{\Gamma(p+1)} [\mathfrak{K}^{**} ||\mathbb{X}||^q + M_*][t_1 - t_2]. \end{aligned} \tag{4.10}$$

Hence by Arzela-Ascoli theorem, we claim relatively compactness of the operator $\mathtt{T}(\mathbb{S})$.
Step IV:

$$Let \quad \mathbb{X} \in \mathbf{E} = \{\mathbb{X} \in B : \mathbb{X} = m_1 \mathtt{T}\mathbb{X},\ \ m_1 \in (0,1)\}, \tag{4.11}$$

and $t \in J$, we have

$$||\mathbb{X}|| = m_1 ||\mathtt{T}\mathbb{X}|| \leq m_1 \left[|\mathbb{X}_0| + \frac{T^p}{\Gamma(p+1)} []\mathfrak{K}^{**} ||\mathbb{X}||^q + M_* \right]. \tag{4.12}$$

This clarifies that the set **E** is bounded. Since all the axioms of Schaefer's fixed point theorem satisfied so $\mathtt{T}$ has minimum of one fixed point, and as a result, the proposed problem (4.3) has at least one solution. $\blacksquare$

Remark 4.2. *If the supposition (P1) is developed for $q = 1$, then there will be no impact on the conclusion of theorem* (4.1)*, if* $\frac{T^p \mathfrak{K}^{**}}{\Gamma(p+1)} < 1$.

Theorem 4.3. *The problem* (4.3) *has one and only one solution, if* $\frac{T^p \mathfrak{K}^{**}}{\Gamma(p+1)} < 1$.

Proof. With the help of the Banach contraction theorem, assume that $\mathbb{X}, \bar{\mathbb{X}} \in X$, then

$$\begin{aligned} ||\mathtt{T}\mathbb{X} - \mathtt{T}\bar{\mathbb{X}}|| &\leq \max_{t\in J} \frac{1}{p} \int_0^t \Big|(\mathrm{t}-\xi)^{p-1}\Big| |\Psi(\xi, \mathbb{X}(\xi)) - \Psi(\xi, \bar{\mathbb{X}}(\xi))| d\xi, \\ &\leq \frac{T^p L_\Psi}{\Gamma(p+1)} ||\mathbb{X} - \bar{\mathbb{X}}||. \end{aligned} \tag{4.13}$$

Therefore $\mathtt{T}$ has one and only one fixed point and as a result, the problem (4.3), we studied in this paper, has also one and only one solution. ∎

5 Stability Analysis

This portion of paper is committed to the functional stability of proposed model. We will look at different varieties of functional stability in this situation

Definition 5.1. *if for $\epsilon > 0$ and Let $\mathbb{X}$ be any solution of the inequality given by*

$$||\mathbb{X} - \mathtt{T}\mathbb{X}|| \leq \epsilon, \ \forall \ \ t \in J, \tag{5.1}$$

The solution of equation (4.3) is Ulam-Hyer stable, if there exist a solution $\bar{\mathbb{X}}$ for the equation (4.3) with the occurrence of a constant $C_q > 0$ and satisfy

$$||\bar{\mathbb{X}} - \mathbb{X}|| \leq \epsilon, \ \forall \ \ t \in J. \tag{5.2}$$

And the solution will be Generalized UHS, if for a positive function $\theta \in C(R, R)$ with $\theta(0) = 0$, we have

$$||\bar{\mathbb{X}} - \mathbb{X}|| \leq C_q \theta(t), \tag{5.3}$$

Remark 5.2. *$\mathbb{X} \in B$ will satisfy (5.1) if$\exists$ $\mathfrak{W}(\mathrm{t}) \in C(J, R)$, such that*
(i) $|\mathfrak{W}(\mathrm{t})| \leq \epsilon, \ \ \forall \ t \in J,$
(ii) ${}^C D^p_{+0} \mathbb{X}(\mathrm{t}) = \Psi(\mathrm{t}, \mathbb{X}(\mathrm{t})) + \mathfrak{W}(\mathrm{t}), \ \ \forall t \in J.$

Lemma 5.3. *Let $\mathbb{X}(\mathrm{t})$ is the solution of the perturbed equation*

$$\begin{cases} {}^C D^p_{+0} \mathbb{X}(\mathrm{t}) = \Psi(\mathrm{t}, \mathbb{X}(\mathrm{t})) + \mathfrak{W}(\mathrm{t}), \\ \mathbb{X}(0) = \mathbb{X}_0. \end{cases} \tag{5.4}$$

then the relation given below will be satisfied.

$$|\mathbb{X}(\mathrm{t}) - \mathtt{T}\mathbb{X}(\mathrm{t})| \leq a\epsilon, \ \ where \ \ a = \frac{T^p}{\Gamma(p+1)} \tag{5.5}$$

Theorem 5.4. *By using Lemma 5.3, the solution of the presented problem in this article (4.3) is UH stable and also GUH stable, if $\frac{T^p L_\omega}{\Gamma(p+1)} < 1$.*

Proof. Let $\mathbb{X} \in X$ and $\bar{\mathbb{X}} \in X$ be the approximate and exact solutions of equation (4.3), then

$$\begin{aligned} |\mathbb{X}(\mathrm{t}) - \bar{\mathbb{X}}(\mathrm{t})| &= |\mathbb{X}(\mathrm{t}) - \mathtt{T}\bar{\mathbb{X}}(\mathrm{t})|, \\ &\leq |\mathbb{X}(\mathrm{t}) - \mathtt{T}\mathbb{X}(\mathrm{t})| + |\mathtt{T}\mathbb{X}(\mathrm{t}) + \mathtt{T}\bar{\mathbb{X}}(\mathrm{t})|, \\ &\leq a\epsilon + \frac{T^p L_\phi}{\Gamma(p+1)} |\mathbb{X}(\mathrm{t}) - \bar{\mathbb{X}}(\mathrm{t})|, \\ &\leq \frac{a\epsilon}{1 - \frac{T^p L_\theta}{\Gamma(p+1)}}. \end{aligned} \tag{5.6}$$

Which clarifies that the studied problem (4.3) is UH stable, also it is GUH stable, by considering

$$Y(\epsilon) = \frac{a\epsilon}{1 - \frac{T^p L_\theta}{\Gamma(p+1)}}. \tag{5.7}$$

$\ni Y(0) = 0$. ∎

Definition 5.5. *The solution of the fractional Equation (2.2) is UH Rassais stable with regard to a function $g \in C(J, R)$, if for $K_q > 0$ and every solution $\mathbb{X} \in B$ of*

$$||^C D^p_{+0}\mathbb{X}(\mathrm{t}) = \Psi(\mathrm{t}, \mathbb{X}(\mathrm{t}))|| \leq g(\mathrm{t})\epsilon, \tag{5.8}$$

$\exists$ a solution $\bar{\mathbb{X}}$ for (2.2) such that

$$||\bar{\mathbb{X}} - \mathbb{X}|| \leq K_q g(\mathrm{t})\epsilon, \ \ \forall \ t \in J. \tag{5.9}$$

For the specific value of $\epsilon = 1$ in above definition, the solution is then called generalized UHR stale.

Remark 5.6. *If $\exists\, \mathfrak{W}(\mathrm{t}) \in C(J, R)$, then $\bar{\mathbb{X}} \in X$ satisfies (5.1), if*
(i) $|\mathfrak{W}(\mathrm{t})| \leq \epsilon g(\mathrm{t}), \ \ \forall \ t \in J,$
(ii) ${}^C D^p_{+0}\mathbb{X}(\mathrm{t}) = \Psi(\mathrm{t}, \mathbb{X}(\mathrm{t})) + \mathfrak{W}(\mathrm{t}), \ \ \forall \ t \in J.$

Lemma 5.7. *Equation (5.4) holds for the result stated below*

$$|\mathbb{X}(\mathrm{t}) - \mathrm{T}\mathbb{X}(\mathrm{t})| \leq a g(\mathrm{t})\epsilon, \ \ \textit{where} \ \ a = \frac{T^p}{\Gamma(p+1)} \tag{5.10}$$

Theorem 5.8. *By lemma (5.7), the result of the studied problem (4.3) is UHR stable & Generalized UHR stable also, if $\frac{T^p L_\phi}{\Gamma p+1} < 1$.*

Proof. Let $\mathbb{X} \in X$ be an approximate solution and $\bar{\mathbb{X}} \in X$ be the exact solution for the equation (4.3), then

$$\begin{aligned}
|\mathbb{X}(\mathrm{t}) - \bar{\mathbb{X}}(\mathrm{t})| &= |\mathbb{X}(\mathrm{t}) - \mathrm{T}\bar{\mathbb{X}}(\mathrm{t})|, \\
&\leq |\mathbb{X}(\mathrm{t}) - \mathrm{T}\mathbb{X}(\mathrm{t})| + |\mathrm{T}\mathbb{X}(\mathrm{t}) + \mathrm{T}\bar{\mathbb{X}}(\mathrm{t})|, \\
&\leq a.g(\mathrm{t})\epsilon + \frac{T^p L_\phi}{\Gamma(p+1)}|\mathbb{X}(\mathrm{t}) - \bar{\mathbb{X}}(\mathrm{t})|, \\
&\leq \frac{a.g(\mathrm{t})\epsilon}{1 - \frac{T^p L_\theta}{\Gamma(p+1)}}.
\end{aligned} \tag{5.11}$$

Therefore, the equation (4.3) is UHR stable, and also generalized UHR stable. ∎

6 Numerical Solution

This section of the paper is devoted to the numerical solution of the model consisting of three FDEs having Caputo's fractional derivative of order p. For the numerical solution we have used FABs presented in [22]. The considered model is

$$\begin{aligned}
{}^c D^p S(\mathrm{t}) &= \Lambda - \frac{\varphi S(\mathrm{t})M(\mathrm{t})}{N} - \frac{\varphi S(\mathrm{t})R(\mathrm{t})}{N} + \frac{\delta S(\mathrm{t})R(\mathrm{t})}{N} - \eta S(\mathrm{t}), \\
{}^c D^p M(\mathrm{t}) &= \frac{\varphi S(\mathrm{t})M(\mathrm{t})}{N} + \frac{\varphi S(\mathrm{t})R(\mathrm{t})}{N} - \frac{\gamma M(\mathrm{t})R(\mathrm{t})}{N} - (\eta + \mu)M(\mathrm{t})), \\
{}^c D^p R(\mathrm{t}) &= \mu M(\mathrm{t}) + \frac{\gamma M(\mathrm{t})R(\mathrm{t})}{N} - \frac{\delta S(\mathrm{t})R(\mathrm{t})}{N} - \eta R(\mathrm{t}).
\end{aligned} \tag{6.1}$$

By using the fundamental theorem of fractional calculus and taking $t = t_{n+1}$ and $t = t_n$ and making use of the lagrange polynomial we get the following result.

$$\begin{aligned}
S_{n+1} &= S(\mathrm{t}_n) + \frac{\mathfrak{F}_1 S(\mathrm{t}_n, S_n)}{i\Gamma(p)}\left\{\frac{2i}{p}t_{n+1}^p - \frac{t_{n+1}^{p+1}}{p+1} + \frac{i}{p}t_n^p - \frac{t_n^{p+1}}{p}\right\} \\
&+ \frac{\mathfrak{F}_1 S(\mathrm{t}_{n-1}, S_{n-1})}{i\Gamma(p)}\left\{\frac{i}{p}t_{n+1}^p - \frac{t_{n+1}^{p+1}}{p+1} + \frac{t_n^p}{p+1}\right\}, \\
M_{n+1} &= M(\mathrm{t}_n) + \frac{\mathfrak{F}_2 M(\mathrm{t}_n, M_n)}{i\Gamma(p)}\left\{\frac{2i}{p}t_{n+1}^p - \frac{t_{n+1}^{p+1}}{p+1} + \frac{i}{p}t_n^p - \frac{t_n^{p+1}}{p}\right\} \\
&+ \frac{\mathfrak{F}_2 M(\mathrm{t}_{n-1}, M_{n-1})}{i\Gamma(p)}\left\{\frac{i}{p}t_{n+1}^p - \frac{t_{n+1}^{p+1}}{p+1} + \frac{t_n^p}{p+1}\right\}, \\
R_{n+1} &= R(\mathrm{t}_n) + \frac{\mathfrak{F}_3 R(\mathrm{t}_n, R_n)}{i\Gamma(p)}\left\{\frac{2i}{p}t_{n+1}^p - \frac{t_{n+1}^{p+1}}{p+1} + \frac{i}{p}t_n^p - \frac{t_n^{p+1}}{p}\right\} \\
&+ \frac{\mathfrak{F}_3 R(\mathrm{t}_{n-1}, R_{n-1})}{i\Gamma(p)}\left\{\frac{i}{p}t_{n+1}^p - \frac{t_{n+1}^{p+1}}{p+1} + \frac{t_n^p}{p+1}\right\}.
\end{aligned} \tag{6.2}$$

Where

$$\begin{aligned}
\mathfrak{F}_1 &= \Lambda - \frac{\varphi S(\mathrm{t})M(\mathrm{t})}{N} - \frac{\varphi S(\mathrm{t})R(\mathrm{t})}{N} + \frac{\delta S(\mathrm{t})R(\mathrm{t})}{N} - \eta S(\mathrm{t}), \\
\mathfrak{F}_2 &= \frac{\varphi S(\mathrm{t})M(\mathrm{t})}{N} + \frac{\varphi S(\mathrm{t})R(\mathrm{t})}{N} - \frac{\gamma M(\mathrm{t})R(\mathrm{t})}{N} - (\eta + \mu)M(\mathrm{t})), \\
\mathfrak{F}_3 &= \mu M(\mathrm{t}) + \frac{\gamma M(\mathrm{t})R(\mathrm{t})}{N} - \frac{\delta S(\mathrm{t})R(\mathrm{t})}{N} - \eta R(\mathrm{t}).
\end{aligned}$$

7 Numerical simulations

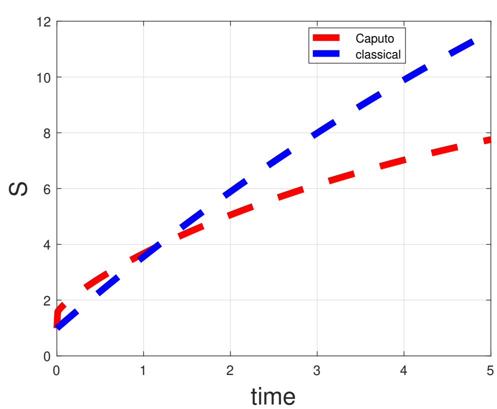
This section of the article is devoted to the simulation of the numerical results obtained in Section 6. For the simulations of the results, the parameters of the model had assigned the values given in (2.2) in Table (1).

Parameters	Values	Source
Λ	2.8	Supposed
φ	0.07	Supposed
δ	0.07	Supposed
η	0.10	Supposed
μ	0.10	Supposed
γ	0.15	Supposed
$S(0)$	0.99	Supposed
$M(0)$	0.01	Supposed
$R(0)$	0.00	Supposed

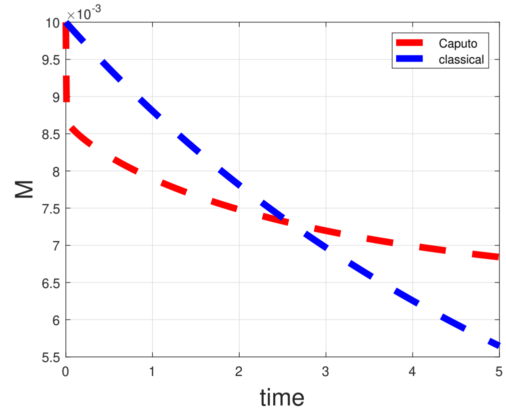
Table 1: Parameters values.

The numerical values of the parameters and compartments of the model we study in this paper are given in (1) while the time domain for the simulation has been restricted to $0 - 500$. With the help of

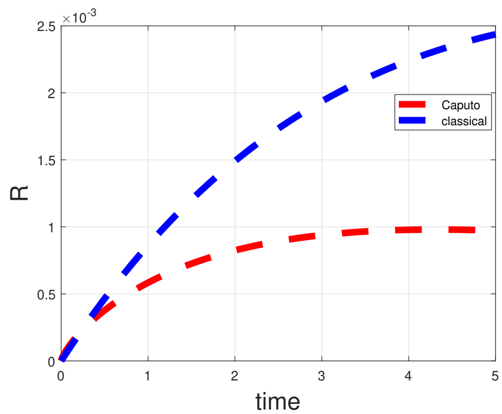
this information we visualizes the numerical results presented in the figures given below. The parameters of the model are expressed in Table (1). The time interval for the simulation of the results was considered as $0 - 500$. The graphical results are presented under the caption Figure 1 and Figure 2. The figures from Fig:1-Fig:3 show the comparison between the behavior of the system while considering the classical order derivative and the fractional order Caputo's derivative.
For the sake of the simulation of the results obtained in (6.2), the authors have assigned different values to the order of the Caputo's derivative and obtained different curves for the corresponding order as shown in the Figure 2. The representation of each figure is given at the bottom of the corresponding figure.



(a) $S(\mathrm{t})$−Non consumer individuals

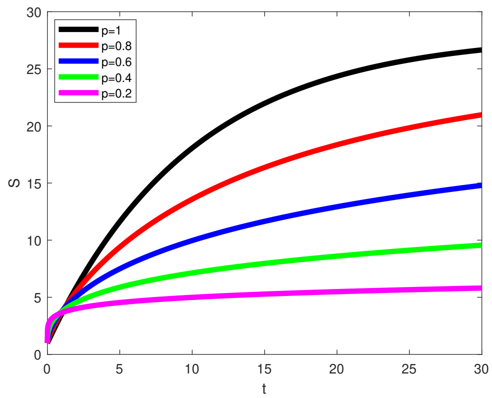


(b) $M(\mathrm{t})$−Nonrisk consumers

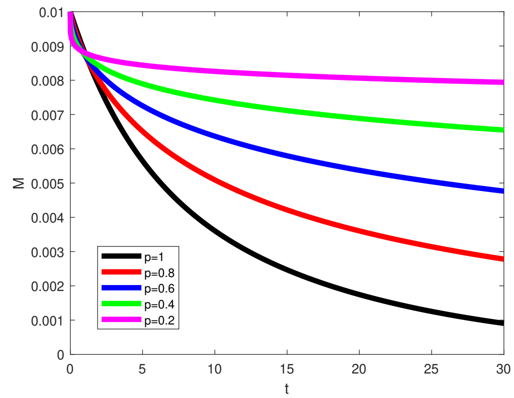


(c) $R(\mathrm{t})$−Risk consumers

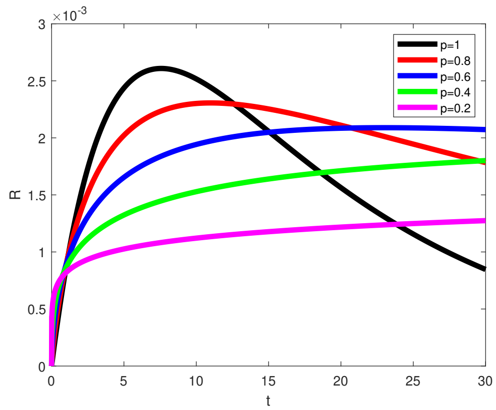
Figure 1: Simulations of $S(\mathrm{t}), M(\mathrm{t}), R(\mathrm{t})$ visuals for the deterministic version of model (2.2) with Caputo versus each of the state variables of model.



(a) $S(\mathrm{t})$—Non consumer individuals



(b) $M(\mathrm{t})$—Nonrisk consumers



(c) $R(\mathrm{t})$—Risk consumers

Figure 2: Graphical behavior of $S(\mathrm{t}), M(\mathrm{t}), R(\mathrm{t})$ individuals for the states variables with different fractional values of model (2.2).

8 Concluding remarks

In this research article, the qualitative and quantitative study of fractional order alcohol drinking model is carried out. For the qualitative study, the authors have developed the conditions for the existence, uniqueness and stability analysis of the considered model. For quantitative analysis, the model has been solved numerically by a convergent and stable fractional Adam's Bashforth scheme and a general scheme of numerical solution is constructed. For the validity of the procedure, a test example is also provided which shows the authenticity of procedure. The results obtained by the numerical scheme are simulated through Matlab which are presented in Section 8.

Author contributions

All authors has equally contributed to the preparation/drafting of this paper.

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Conflict of interests

This work does not have any potential conflicts of interest.

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Declaration Statement of Generative AI

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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